

Elementary Linear Algebra

8e

Ron Larson

The Pennsylvania State University
The Behrend College



Australia • Brazil • Mexico • Singapore • United Kingdom • United States

Elementary Linear Algebra
Eighth Edition**Ron Larson**

Product Director: Terry Boyle
Product Manager: Richard Stratton
Content Developer: Spencer Arritt
Product Assistant: Kathryn Schrupf
Marketing Manager: Ana Albinson
Content Project Manager: Jennifer Riden
Manufacturing Planner: Doug Bertke
Production Service: Larson Texts, Inc.
Photo Researcher: Lumina Datamatics
Text Researcher: Lumina Datamatics
Text Designer: Larson Texts, Inc.
Cover Designer: Larson Texts, Inc.
Cover Image: Keo/Shutterstock.com
Compositor: Larson Texts, Inc.

© 2017, 2013, 2009 Cengage Learning

WCN: 02-200-203

ALL RIGHTS RESERVED. No part of this work covered by the copyright herein may be reproduced, transmitted, stored, or used in any form or by any means graphic, electronic, or mechanical, including but not limited to photocopying, recording, scanning, digitizing, taping, Web distribution, information networks, or information storage and retrieval systems, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without the prior written permission of the publisher.

For product information and technology assistance, contact us at
Cengage Learning Customer & Sales Support, 1-800-354-9706.

For permission to use material from this text or product,
submit all requests online at **www.cengage.com/permissions**.
Further permissions questions can be e-mailed to
permissionrequest@cengage.com.

Library of Congress Control Number: 2015944033

Student Edition

ISBN: 978-1-305-65800-4

Loose-leaf Edition

ISBN: 978-1-305-95320-8

Cengage Learning

20 Channel Center Street
Boston, MA 02210
USA

Cengage Learning is a leading provider of customized learning solutions with employees residing in nearly 40 different countries and sales in more than 125 countries around the world. Find your local representative at **www.cengage.com**.

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Cengage Learning Solutions, visit **www.cengage.com**.
Purchase any of our products at your local college store or at our preferred online store **www.cengagebrain.com**.

Contents

1	■	Systems of Linear Equations	1
1.1		Introduction to Systems of Linear Equations	2
1.2		Gaussian Elimination and Gauss-Jordan Elimination	13
1.3		Applications of Systems of Linear Equations	25
		<i>Review Exercises</i>	35
		<i>Project 1 Graphing Linear Equations</i>	38
		<i>Project 2 Underdetermined and Overdetermined Systems</i>	38
2	■	Matrices	39
2.1		Operations with Matrices	40
2.2		Properties of Matrix Operations	52
2.3		The Inverse of a Matrix	62
2.4		Elementary Matrices	74
2.5		Markov Chains	84
2.6		More Applications of Matrix Operations	94
		<i>Review Exercises</i>	104
		<i>Project 1 Exploring Matrix Multiplication</i>	108
		<i>Project 2 Nilpotent Matrices</i>	108
3	■	Determinants	109
3.1		The Determinant of a Matrix	110
3.2		Determinants and Elementary Operations	118
3.3		Properties of Determinants	126
3.4		Applications of Determinants	134
		<i>Review Exercises</i>	144
		<i>Project 1 Stochastic Matrices</i>	147
		<i>Project 2 The Cayley-Hamilton Theorem</i>	147
		<i>Cumulative Test for Chapters 1–3</i>	149
4	■	Vector Spaces	151
4.1		Vectors in R^n	152
4.2		Vector Spaces	161
4.3		Subspaces of Vector Spaces	168
4.4		Spanning Sets and Linear Independence	175
4.5		Basis and Dimension	186
4.6		Rank of a Matrix and Systems of Linear Equations	195
4.7		Coordinates and Change of Basis	208
4.8		Applications of Vector Spaces	218
		<i>Review Exercises</i>	227
		<i>Project 1 Solutions of Linear Systems</i>	230
		<i>Project 2 Direct Sum</i>	230

5	■ Inner Product Spaces	231
5.1	Length and Dot Product in R^n	232
5.2	Inner Product Spaces	243
5.3	Orthonormal Bases: Gram-Schmidt Process	254
5.4	Mathematical Models and Least Squares Analysis	265
5.5	Applications of Inner Product Spaces	277
	<i>Review Exercises</i>	290
	<i>Project 1 The QR-Factorization</i>	293
	<i>Project 2 Orthogonal Matrices and Change of Basis</i>	294
	<i>Cumulative Test for Chapters 4 and 5</i>	295
6	■ Linear Transformations	297
6.1	Introduction to Linear Transformations	298
6.2	The Kernel and Range of a Linear Transformation	309
6.3	Matrices for Linear Transformations	320
6.4	Transition Matrices and Similarity	330
6.5	Applications of Linear Transformations	336
	<i>Review Exercises</i>	343
	<i>Project 1 Reflections in R^2 (I)</i>	346
	<i>Project 2 Reflections in R^2 (II)</i>	346
7	■ Eigenvalues and Eigenvectors	347
7.1	Eigenvalues and Eigenvectors	348
7.2	Diagonalization	359
7.3	Symmetric Matrices and Orthogonal Diagonalization	368
7.4	Applications of Eigenvalues and Eigenvectors	378
	<i>Review Exercises</i>	393
	<i>Project 1 Population Growth and Dynamical Systems (I)</i>	396
	<i>Project 2 The Fibonacci Sequence</i>	396
	<i>Cumulative Test for Chapters 6 and 7</i>	397
8	■ Complex Vector Spaces (online)*	
8.1	Complex Numbers	
8.2	Conjugates and Division of Complex Numbers	
8.3	Polar Form and DeMoivre's Theorem	
8.4	Complex Vector Spaces and Inner Products	
8.5	Unitary and Hermitian Matrices	
	<i>Review Exercises</i>	
	<i>Project 1 The Mandelbrot Set</i>	
	<i>Project 2 Population Growth and Dynamical Systems (II)</i>	

9 **Linear Programming (online)***

- 9.1** Systems of Linear Inequalities
- 9.2** Linear Programming Involving Two Variables
- 9.3** The Simplex Method: Maximization
- 9.4** The Simplex Method: Minimization
- 9.5** The Simplex Method: Mixed Constraints

Review Exercises

Project 1 Beach Sand Replenishment (I)

Project 2 Beach Sand Replenishment (II)

10 **Numerical Methods (online)***

- 10.1** Gaussian Elimination with Partial Pivoting
- 10.2** Iterative Methods for Solving Linear Systems
- 10.3** Power Method for Approximating Eigenvalues
- 10.4** Applications of Numerical Methods

Review Exercises

Project 1 The Successive Over-Relaxation (SOR) Method

Project 2 United States Population

Appendix

A1

Mathematical Induction and Other Forms of Proofs

Answers to Odd-Numbered Exercises and Tests

A7

Index

A41

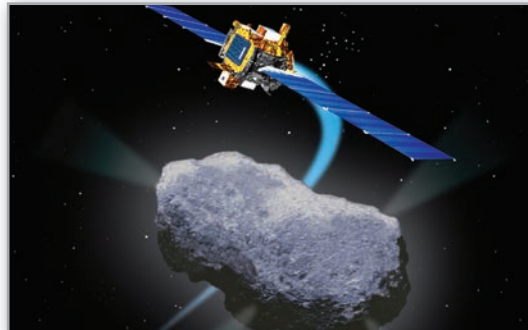
Technology Guide*

*Available online at **CengageBrain.com**.

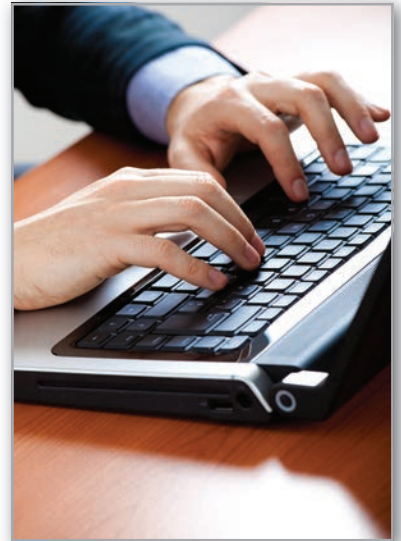
3 Determinants



- 3.1 The Determinant of a Matrix
- 3.2 Determinants and Elementary Operations
- 3.3 Properties of Determinants
- 3.4 Applications of Determinants



Comet Landing (p. 141)



Software Publishing (p. 143)



Engineering and Control (p. 130)



Sudoku (p. 120)



Volume of a Tetrahedron (p. 114)

3.1 The Determinant of a Matrix

-  Find the determinant of a 2×2 matrix.
-  Find the minors and cofactors of a matrix.
-  Use expansion by cofactors to find the determinant of a matrix.
-  Find the determinant of a triangular matrix.

THE DETERMINANT OF A 2×2 MATRIX

Every *square* matrix can be associated with a real number called its *determinant*. Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations. For example, the system

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2\end{aligned}$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$. (See Exercise 53.) Note that both fractions have the same denominator, $a_{11}a_{22} - a_{21}a_{12}$. This quantity is the *determinant* of the coefficient matrix of the system.

REMARK

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A . Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended. Furthermore, it is common practice to delete the matrix brackets and write

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

instead of

$$\left[\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right].$$

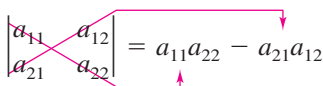
Definition of the Determinant of a 2×2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{is } \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

The diagram below shows a convenient method for remembering the formula for the determinant of a 2×2 matrix.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$


The determinant is the difference of the products of the two diagonals of the matrix. Note that the order of the products is important.

REMARK

Notice that the determinant of a matrix can be positive, zero, or negative.

EXAMPLE 1

Determinants of Matrices of Order 2

- a. For $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$.
- b. For $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$.
- c. For $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$, $|C| = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2(\frac{3}{2}) = 0 - 3 = -3$.



MINORS AND COFACTORS

To define the determinant of a square matrix of order higher than 2, it is convenient to use *minors* and *cofactors*.

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the i th row and j th column of A . The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

The minors and cofactors of a matrix can differ only in sign. To obtain the cofactors of a matrix, first find the minors and then apply the checkerboard pattern of $+$'s and $-$'s shown at the left. Note that *odd* positions (where $i + j$ is odd) have negative signs, and even positions (where $i + j$ is even) have positive signs.

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$n \times n$ matrix

EXAMPLE 2

Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

SOLUTION

To find the minor M_{11} , delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Verify that the minors are

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6. \end{array}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a 3×3 matrix shown above.

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6 \end{array}$$

REMARK

The determinant of a matrix of order 1 is simply the entry of the matrix. For example, if $A = [-2]$, then

$$\det(A) = -2.$$

THE DETERMINANT OF A SQUARE MATRIX

The definition below is **inductive** because it uses the determinant of a square matrix of order $n - 1$ to define the determinant of a square matrix of order n .

Definition of the Determinant of a Square Matrix

If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^n a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Confirm that, for 2×2 matrices, this definition yields

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

as previously defined.

When you use this definition to evaluate a determinant, you are **expanding by cofactors in the first row**. Example 3 demonstrates this procedure.

EXAMPLE 3**The Determinant of a Matrix of Order 3**

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

SOLUTION

This is the same matrix as in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad C_{13} = 4.$$

So, by the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding in *any* row or column. For instance, you could expand the matrix in Example 3 in the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14 \end{aligned}$$

or in the first column to obtain

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} && \text{First column expansion} \\ &= 0(-1) + 3(-2) + 4(5) \\ &= 14. \end{aligned}$$

Try other possibilities to confirm that the determinant of A can be evaluated by expanding in *any* row or column. The theorem on the next page states this, and is known as Laplace's Expansion of a Determinant, after the French mathematician Pierre Simon de Laplace (1749–1827).

THEOREM 3.1 Expansion by Cofactors

Let A be a square matrix of order n . Then the determinant of A is

$$\det(A) = |A| = \sum_{j=1}^n a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{\textit{ith row expansion}}$$

or

$$\det(A) = |A| = \sum_{i=1}^n a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \quad \text{\textit{jth column expansion}}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$\begin{aligned} a_{ij}C_{ij} &= (0)C_{ij} \\ &= 0 \end{aligned}$$

The row (or column) containing the most zeros is usually the best choice for expansion by cofactors. The next example demonstrates this.

EXAMPLE 4**The Determinant of a Matrix of Order 4**

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

SOLUTION

Notice that three of the entries in the third column are zeros. So, to eliminate some of the work in the expansion, use the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

The cofactors C_{23} , C_{33} , and C_{43} have zero coefficients, so you need only find the cofactor C_{13} . To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \quad \text{\textit{Delete 1st row and 3rd column.}}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \quad \text{\textit{Simplify.}}$$

Expanding by cofactors in the second row yields

$$\begin{aligned} C_{13} &= (0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \\ &= 0 + 2(1)(-4) + 3(-1)(-7) \\ &= 13. \end{aligned}$$

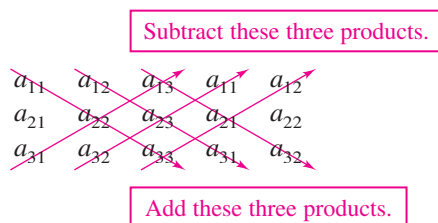
You obtain

$$\begin{aligned} |A| &= 3(13) \\ &= 39. \end{aligned}$$

TECHNOLOGY

Many graphing utilities and software programs can find the determinant of a square matrix. If you use a graphing utility, then you may see something similar to the screen below for Example 4. The **Technology Guide** at *CengageBrain.com* can help you use technology to find a determinant.

An alternative method is commonly used to evaluate the determinant of a 3×3 matrix A . To apply this method, copy the first and second columns of A to form fourth and fifth columns. Then obtain the determinant of A by adding (or subtracting) the products of the six diagonals, as shown in the diagram below.



Confirm that the determinant of A is

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

EXAMPLE 5

The Determinant of a Matrix of Order 3

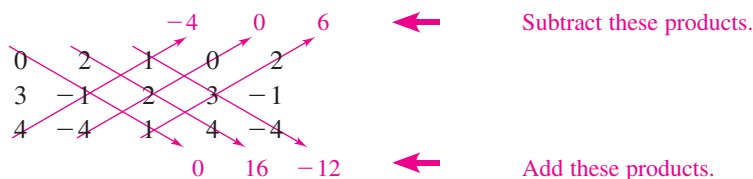
See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}.$$

SOLUTION

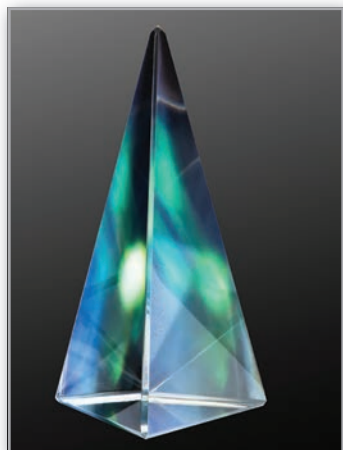
Begin by copying the first two columns and then computing the six diagonal products as shown below.



Now, by adding the lower three products and subtracting the upper three products, you can find the determinant of A to be

$$|A| = 0 + 16 + (-12) - (-4) - 0 - 6 = 2.$$

The diagonal process illustrated in Example 5 is valid *only* for matrices of order 3. For matrices of higher order, you must use another method.



LINEAR ALGEBRA APPLIED

Recall that a **tetrahedron** is a polyhedron consisting of four triangular faces. One practical application of determinants is in finding the volume of a tetrahedron in a coordinate plane. If the vertices of a tetrahedron are (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and (x_4, y_4, z_4) , then the volume is

$$\text{Volume} = \pm \frac{1}{6} \det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}.$$

You will study this and other applications of determinants in Section 3.4.

magnetix/Shutterstock.com

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Recall from Section 2.4 that a square matrix is *upper triangular* when it has all zero entries below its main diagonal, and *lower triangular* when it has all zero entries above its main diagonal, as shown in the diagram at the left. A matrix that is both upper and lower triangular is a **diagonal matrix**. That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

To find the determinant of a triangular matrix, simply form the product of the entries on the main diagonal. It should be easy to see that this procedure is valid for triangular matrices of order 2 or 3. For example, to find the determinant of

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

expand in the third row to obtain

$$\begin{aligned} |A| &= 0(-1)^{3+1} \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 0(-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + 3(-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \\ &= 3(1)(-2) \\ &= -6 \end{aligned}$$

which is the product of the entries on the main diagonal.

THEOREM 3.2 Determinant of a Triangular Matrix

If A is a triangular matrix of order n , then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}.$$

PROOF

Use *mathematical induction** to prove this theorem for the case in which A is an upper triangular matrix. The proof of the case in which A is lower triangular is similar. If A has order 1, then $A = [a_{11}]$ and the determinant is $|A| = a_{11}$. Assuming the theorem is true for any upper triangular matrix of order $k-1$, consider an upper triangular matrix A of order k . Expanding in the k th row, you obtain

$$|A| = 0C_{k1} + 0C_{k2} + \cdots + 0C_{k(k-1)} + a_{kk}C_{kk} = a_{kk}C_{kk}.$$

Now, note that $C_{kk} = (-1)^{2k}M_{kk} = M_{kk}$, where M_{kk} is the determinant of the upper triangular matrix formed by deleting the k th row and k th column of A . This matrix is of order $k-1$, so apply the induction assumption to write

$$|A| = a_{kk}M_{kk} = a_{kk}(a_{11}a_{22}a_{33} \cdots a_{k-1,k-1}) = a_{11}a_{22}a_{33} \cdots a_{kk}.$$

EXAMPLE 6**The Determinant of a Triangular Matrix**

The determinant of the lower triangular matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

is $|A| = (2)(-2)(1)(3) = -12$.

*See Appendix for a discussion of mathematical induction.

3.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Determinant of a Matrix** In Exercises 1–12, find the determinant of the matrix.

1. $[1]$
2. $[-3]$
3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$
4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$
5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$
6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$
7. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$
8. $\begin{bmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{bmatrix}$
9. $\begin{bmatrix} 0 & 8 \\ 0 & 4 \end{bmatrix}$
10. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$
11. $\begin{bmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{bmatrix}$
12. $\begin{bmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix In Exercises 13–16, find all (a) minors and (b) cofactors of the matrix.

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
14. $\begin{bmatrix} -5 & 6 \\ 1 & 0 \end{bmatrix}$
15. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$
16. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

17. Find the determinant of the matrix in Exercise 15 using the method of expansion by cofactors. Use (a) the second row and (b) the second column.
18. Find the determinant of the matrix in Exercise 16 using the method of expansion by cofactors. Use (a) the third row and (b) the first column.

Finding a Determinant In Exercises 19–32, use expansion by cofactors to find the determinant of the matrix.

19. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$
20. $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}$
21. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$
22. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$
23. $\begin{bmatrix} -0.4 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$
24. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$
25. $\begin{bmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
26. $\begin{bmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$

$$27. \begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix} \quad 28. \begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{bmatrix}$$

$$29. \begin{bmatrix} w & x & y & z \\ 21 & -15 & 24 & 30 \\ -10 & 24 & -32 & 18 \\ -40 & 22 & 32 & -35 \end{bmatrix}$$

$$30. \begin{bmatrix} w & x & y & z \\ 10 & 15 & -25 & 30 \\ -30 & 20 & -15 & -10 \\ 30 & 35 & -25 & -40 \end{bmatrix}$$

$$31. \begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$32. \begin{bmatrix} -4 & 3 & 2 & -1 & -2 \\ 1 & -2 & 7 & -13 & -12 \\ -6 & 2 & -5 & -6 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & -2 & 0 & -9 \end{bmatrix}$$

Finding a Determinant In Exercises 33 and 34, use the method demonstrated in Example 5 to find the determinant of the matrix.

$$33. \begin{bmatrix} 3 & 0 & 4 \\ -2 & 4 & 1 \\ 1 & -3 & 1 \end{bmatrix} \quad 34. \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{bmatrix}$$

**Finding a Determinant** In Exercises 35–38, use a software program or a graphing utility to find the determinant of the matrix.

$$35. \begin{bmatrix} 0.1 & 0.6 & -0.3 \\ 0.7 & -0.1 & 0.1 \\ 0.1 & 0.3 & -0.8 \end{bmatrix} \quad 36. \begin{bmatrix} 4 & 3 & 2 & 5 \\ 1 & 6 & -1 & 2 \\ -3 & 2 & 4 & 5 \\ 6 & 1 & 3 & -2 \end{bmatrix}$$



$$37. \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$$



$$38. \begin{bmatrix} 8 & 5 & 1 & -2 & 0 \\ -1 & 0 & 7 & 1 & 6 \\ 0 & 8 & 6 & 5 & -3 \\ 1 & 2 & 5 & -8 & 4 \\ 2 & 6 & -2 & 0 & 6 \end{bmatrix}$$

Finding the Determinant of a Triangular Matrix In Exercises 39–42, find the determinant of the triangular matrix.

39.
$$\begin{bmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{bmatrix}$$

40.
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

41.
$$\begin{bmatrix} 5 & 8 & -4 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

42.
$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{bmatrix}$$

True or False? In Exercises 43 and 44, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

43. (a) The determinant of a 2×2 matrix A is $a_{21}a_{12} - a_{11}a_{22}$.
 (b) The determinant of a matrix of order 1 is the entry of the matrix.
 (c) The ij -cofactor of a square matrix A is the matrix obtained by deleting the i th row and j th column of A .
44. (a) To find the determinant of a triangular matrix, add the entries on the main diagonal.
 (b) To find the determinant of a matrix, expand by cofactors in any row or column.
 (c) When expanding by cofactors, you need not evaluate the cofactors of zero entries.

Solving an Equation In Exercises 45–48, solve for x .

45.
$$\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$$

46.
$$\begin{vmatrix} x-6 & 3 \\ -2 & x+1 \end{vmatrix} = 0$$

47.
$$\begin{vmatrix} x-1 & 2 \\ 3 & x-2 \end{vmatrix} = 0$$

48.
$$\begin{vmatrix} x+3 & 1 \\ -4 & x-1 \end{vmatrix} = 0$$

Solving an Equation In Exercises 49–52, find the values of λ for which the determinant is zero.

49.
$$\begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix}$$

50.
$$\begin{vmatrix} \lambda-5 & 3 \\ 1 & \lambda-5 \end{vmatrix}$$

51.
$$\begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix}$$

52.
$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda-2 \end{vmatrix}$$

53. Show that the system of linear equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

54. CAPSTONE For an $n \times n$ matrix A , explain how to find each value.

- (a) The minor M_{ij} of the entry a_{ij}
 (b) The cofactor C_{ij} of the entry a_{ij}
 (c) The determinant of A

Entries Involving Expressions In Exercises 55–62, evaluate the determinant, in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.

55.
$$\begin{vmatrix} 6u & -1 \\ -1 & 3v \end{vmatrix}$$

56.
$$\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$$

57.
$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

58.
$$\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$$

59.
$$\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$$

60.
$$\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$$

61.
$$\begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

62.
$$\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

Verifying an Equation In Exercises 63–68, evaluate the determinants to verify the equation.

63.
$$\begin{vmatrix} w & x \\ y & z \end{vmatrix} = - \begin{vmatrix} y & z \\ w & x \end{vmatrix}$$

64.
$$\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c \begin{vmatrix} w & x \\ y & z \end{vmatrix}$$

65.
$$\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x+cw \\ y & z+cy \end{vmatrix}$$

66.
$$\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$$

67.
$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

68.
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

69. You are given the equation

$$\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} = ax^2 + bx + c.$$

- (a) Verify the equation.
 (b) Use the equation as a model to find a determinant that is equal to $ax^3 + bx^2 + cx + d$.

70. The determinant of a 2×2 matrix involves two products. The determinant of a 3×3 matrix involves six triple products. Show that the determinant of a 4×4 matrix involves 24 quadruple products.

3.2 Determinants and Elementary Operations

-  Use elementary row operations to evaluate a determinant.
-  Use elementary column operations to evaluate a determinant.
-  Recognize conditions that yield zero determinants.

DETERMINANTS AND ELEMENTARY ROW OPERATIONS

Which of the determinants below is easier to evaluate?

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix} \quad \text{or} \quad |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Given what you know about the determinant of a triangular matrix, it should be clear that the second determinant is *much* easier to evaluate. Its determinant is simply the product of the entries on the main diagonal. That is, $|B| = (1)(2)(-3)(-1) = 6$. Using expansion by cofactors (the only technique discussed so far) to evaluate the first determinant is messy. For example, when you expand by cofactors in the first row, you have

$$|A| = 1 \begin{vmatrix} -6 & 3 & 2 \\ 4 & -9 & -3 \\ -6 & 9 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 3 & 2 \\ -2 & -9 & -3 \\ 3 & 9 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & -6 & 2 \\ -2 & 4 & -3 \\ 3 & -6 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & -6 & 3 \\ -2 & 4 & -9 \\ 3 & -6 & 9 \end{vmatrix}.$$

Evaluating the determinants of these four 3×3 matrices produces

$$|A| = (1)(-60) + (2)(39) + (3)(-10) - (1)(-18) = 6.$$

Note that $|A|$ and $|B|$ have the same value. Also note that you can obtain matrix B from matrix A by adding multiples of the first row to the second, third, and fourth rows. (Verify this.) In this section, you will see the effects of elementary row (and column) operations on the value of a determinant.

EXAMPLE 1

The Effects of Elementary Row Operations on a Determinant

- a. The matrix B is obtained from A by interchanging the rows of A .

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 11 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -11$$

- b. The matrix B is obtained from A by adding -2 times the first row of A to the second row of A .

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} = 2 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2$$

- c. The matrix B is obtained from A by multiplying the first row of A by $\frac{1}{2}$.

$$|A| = \begin{vmatrix} 2 & -8 \\ -2 & 9 \end{vmatrix} = 2 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & -4 \\ -2 & 9 \end{vmatrix} = 1$$

In Example 1, notice that interchanging the two rows of A changes the sign of its determinant, adding -2 times the first row of A to the second row does not change its determinant, and multiplying the first row of A by $\frac{1}{2}$ multiplies its determinant by $\frac{1}{2}$. The next theorem generalizes these observations.

REMARK

Note that the third property enables you to divide a row by the common factor. For example,

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}. \quad \text{Factor 2 out of first row.}$$



Augustin-Louis Cauchy
(1789–1857)

Cauchy's contributions to the study of mathematics were revolutionary, and he is often credited with bringing rigor to modern mathematics. For instance, he was the first to rigorously define limits, continuity, and the convergence of an infinite series. In addition to being known for his work in complex analysis, he contributed to the theories of determinants and differential equations. It is interesting to note that Cauchy's work on determinants preceded Cayley's development of matrices.

**THEOREM 3.3 Elementary Row Operations and Determinants**

Let A and B be square matrices.

1. When B is obtained from A by interchanging two rows of A , $\det(B) = -\det(A)$.
2. When B is obtained from A by adding a multiple of a row of A to another row of A , $\det(B) = \det(A)$.
3. When B is obtained from A by multiplying a row of A by a nonzero constant c , $\det(B) = c \det(A)$.

PROOF

The proof of the first property is below. The proofs of the other two properties are left as exercises. (See Exercises 47 and 48.) Assume that A and B are 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$

Then, you have $|A| = a_{11}a_{22} - a_{21}a_{12}$ and $|B| = a_{21}a_{12} - a_{11}a_{22}$. So $|B| = -|A|$. Using mathematical induction, assume the property is true for matrices of order $(n - 1)$. Let A be an $n \times n$ matrix such that B is obtained from A by interchanging two rows of A . Then, to find $|A|$ and $|B|$, expand in a row other than the two interchanged rows. By the induction assumption, the cofactors of B will be the negatives of the cofactors of A because the corresponding $(n - 1) \times (n - 1)$ matrices have two rows interchanged. Finally, $|B| = -|A|$ and the proof is complete. 

Theorem 3.3 provides a practical way to evaluate determinants. To find the determinant of a matrix A , you can use elementary row operations to obtain a triangular matrix B that is row-equivalent to A . For each step in the elimination process, use Theorem 3.3 to determine the effect of the elementary row operation on the determinant. Finally, find the determinant of B by multiplying the entries on its main diagonal.

EXAMPLE 2**Finding a Determinant Using Elementary Row Operations**

Find the determinant of

$$A = \begin{bmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{bmatrix}.$$

SOLUTION

Using elementary row operations, rewrite A in triangular form as shown below.

$$\begin{aligned} \begin{vmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{vmatrix} &= - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 3 & -8 \end{vmatrix} &\leftarrow \text{Interchange the first two rows.} \\ &= 7 \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 3 & -8 \end{vmatrix} &\leftarrow \text{Factor } -7 \text{ out of the second row.} \\ &= 7 \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{vmatrix} &\leftarrow \text{Add } -3 \text{ times the second row to the third row to produce a new third row.} \end{aligned}$$

The above matrix is triangular, so the determinant is

$$|A| = 7(1)(1)(-2) = -14.$$



DETERMINANTS AND ELEMENTARY COLUMN OPERATIONS

Although Theorem 3.3 is stated in terms of elementary *row* operations, the theorem remains valid when the word “column” replaces the word “row.” Operations performed on the columns (rather than on the rows) of a matrix are **elementary column operations**, and two matrices are **column-equivalent** when one can be obtained from the other by elementary column operations. Here are illustrations of the column versions of Theorem 3.3 Properties 1 and 3.

$$\begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

Interchange the first two columns.

$$\begin{vmatrix} 2 & 3 & -5 \\ 4 & 1 & 0 \\ -2 & 4 & -3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & -5 \\ 2 & 1 & 0 \\ -1 & 4 & -3 \end{vmatrix}$$

Factor 2 out of the first column.

In evaluating a determinant, it is occasionally convenient to use elementary column operations, as shown in Example 3.

EXAMPLE 3

Finding a Determinant Using Elementary Column Operations

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the determinant of $A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix}$.

SOLUTION

The first two columns of A are multiples of each other, so you can obtain a column of zeros by adding 2 times the first column to the second column, as shown below.

$$\begin{vmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & -3 \end{vmatrix}$$

At this point, you do not need to rewrite the matrix in triangular form, because there is an entire column of zeros. Simply conclude that the determinant is zero. The validity of this conclusion follows from Theorem 3.1. Specifically, by expanding by cofactors in the second column, you have

$$|A| = (0)C_{12} + (0)C_{22} + (0)C_{32} = 0.$$



LINEAR ALGEBRA APPLIED

In a Sudoku puzzle, the object is to fill out a partially completed 9×9 grid of boxes with numbers from 1 to 9 so that each column, row, and 3×3 sub-grid contains each number once. For a completed Sudoku grid to be valid, no two rows (or columns) will have the numbers in the same order. If this should happen, then the determinant of the 9×9 matrix formed by the numbers will be zero. This is a direct result of condition 2 of Theorem 3.4 on the next page.

iStockphoto.com/jirkaejc

MATRICES AND ZERO DETERMINANTS

Example 3 shows that when two columns of a matrix are scalar multiples of each other, the determinant of the matrix is zero. This is one of three conditions that yield a determinant of zero.

THEOREM 3.4 Conditions That Yield a Zero Determinant

If A is a square matrix and any one of the conditions below is true, then $\det(A) = 0$.

1. An entire row (or an entire column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

PROOF

Verify each part of this theorem by using elementary row operations and expansion by cofactors. For instance, if an entire row or column consists of zeros, then each cofactor in the expansion is multiplied by zero. When condition 2 or 3 is true, use elementary row or column operations to create an entire row or column of zeros. 

Recognizing the conditions listed in Theorem 3.4 can make evaluating a determinant much easier. For example,

$$\begin{vmatrix} 0 & 0 & 0 \\ 2 & 4 & -5 \\ 3 & -5 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -6 \\ -2 & 0 & 6 \end{vmatrix} = 0.$$



The first row has all zeros.



The first and third rows are the same.



The third column is a multiple of the first column.

Do not conclude, however, that Theorem 3.4 gives the *only* conditions that produce a zero determinant. This theorem is often used indirectly. That is, you may begin with a matrix that does not satisfy any of the conditions of Theorem 3.4 and, through elementary row or column operations, obtain a matrix that does satisfy one of the conditions. Example 4 demonstrates this.

EXAMPLE 4

A Matrix with a Zero Determinant

Find the determinant of

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{bmatrix}.$$

SOLUTION

Adding -2 times the first row to the second row produces

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 4 & 1 \\ 2 + (-2)(1) & -1 + (-2)(4) & 0 + (-2)(1) \\ 0 & 18 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix}. \end{aligned}$$

The second and third rows are multiples of each other, so the determinant is zero. 

In Example 4, you could have obtained a matrix with a row of all zeros by performing an additional elementary row operation (adding 2 times the second row to the third row). This is true in general. That is, a square matrix has a determinant of zero if and only if it is row- (or column-) equivalent to a matrix that has at least one row (or column) consisting entirely of zeros.

You have now studied two methods for evaluating determinants. Of these, the method of using elementary row operations to reduce the matrix to triangular form is usually faster than cofactor expansion along a row or column. If the matrix is large, then the number of arithmetic operations needed for cofactor expansion can become extremely large. For this reason, most computer and calculator algorithms use the method involving elementary row operations. The table below shows the maximum numbers of additions (plus subtractions) and multiplications (plus divisions) needed for each of these two methods for matrices of orders 3, 5, and 10. (Verify this.)

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	8	10
5	119	205	40	44
10	3,628,799	6,235,300	330	339

In fact, the maximum number of additions alone for the cofactor expansion of an $n \times n$ matrix is $n! - 1$. The factorial $30!$ is approximately equal to 2.65×10^{32} , so even a relatively small 30×30 matrix could require an extremely large number of operations. If a computer could do one trillion operations per second, it could still take more than 22 trillion years to compute the determinant of this matrix using cofactor expansion. Yet, row reduction would take only a fraction of a second.

When evaluating a determinant *by hand*, you sometimes save steps by using elementary row (or column) operations to create a row (or column) having zeros in all but one position and then using cofactor expansion to reduce the order of the matrix by 1. The next two examples illustrate this approach.

EXAMPLE 5 Finding a Determinant

Find the determinant of

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}.$$

SOLUTION

Notice that the matrix A already has one zero in the third row. Create another zero in the third row by adding 2 times the first column to the third column, as shown below.

$$|A| = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix}$$

Expanding by cofactors in the third row produces

$$|A| = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -3(-1)^4 \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = -3(1)(-1) = 3.$$

EXAMPLE 6**Finding a Determinant**

Find the determinant of

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}.$$

SOLUTION

The second column of this matrix already has two zeros, so choose it for cofactor expansion. Create two additional zeros in the second column by adding the second row to the fourth row, and then adding -1 times the second row to the fifth row.

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix} \\ &= (1)(-1)^4 \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

You have now reduced the problem of finding the determinant of a 5×5 matrix to the problem of finding the determinant of a 4×4 matrix. The fourth row already has two zeros, so choose it for the next cofactor expansion. Add -3 times the fourth column to the first column.

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ -0 & 0 & 0 & 1 \end{vmatrix} \\ &= (1)(-1)^8 \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \end{aligned}$$

Add the second row to the first row and then expand by cofactors in the first row.

$$\begin{aligned} |A| &= \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} = \begin{vmatrix} -0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \\ &= 5(-1)^4 \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix} \\ &= 5(1)(-27) \\ &= -135 \end{aligned}$$

3.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Properties of Determinants In Exercises 1–20, determine which property of determinants the equation illustrates.

1. $\begin{vmatrix} 2 & -6 \\ 1 & -3 \end{vmatrix} = 0$
2. $\begin{vmatrix} -4 & 5 \\ 12 & -15 \end{vmatrix} = 0$
3. $\begin{vmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 5 & 6 & -7 \end{vmatrix} = 0$
4. $\begin{vmatrix} -3 & 2 & 1 \\ 6 & 0 & 0 \\ -3 & 2 & 1 \end{vmatrix} = 0$
5. $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$
6. $\begin{vmatrix} 1 & 3 & 4 & -5 \\ -2 & 2 & 0 & 1 \\ 1 & 6 & 2 & -7 \\ 0 & 5 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 2 & -7 \\ -2 & 2 & 0 & 1 \\ 1 & 3 & 4 & -5 \\ 0 & 5 & 3 & 8 \end{vmatrix}$
7. $\begin{vmatrix} 5 & 10 \\ 2 & -7 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix}$
8. $\begin{vmatrix} 9 & 1 \\ 3 & 12 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 \\ 1 & 12 \end{vmatrix}$
9. $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12 \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$
10. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & -8 & 6 \\ 5 & 4 & 12 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 \\ 4 & -4 & 2 \\ 5 & 2 & 4 \end{vmatrix}$
11. $\begin{vmatrix} -10 & 5 & 5 \\ 35 & -20 & 25 \\ 0 & 15 & 30 \end{vmatrix} = 5^3 \begin{vmatrix} -2 & 1 & 1 \\ 7 & -4 & 5 \\ 0 & 3 & 6 \end{vmatrix}$
12. $\begin{vmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 6^4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$
13. $\begin{vmatrix} 2 & -3 \\ 8 & 7 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 0 & 19 \end{vmatrix}$
14. $\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$
15. $\begin{vmatrix} 1 & -3 & 2 \\ 5 & 2 & -1 \\ -1 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 2 \\ 0 & 17 & -11 \\ -1 & 0 & 6 \end{vmatrix}$
16. $\begin{vmatrix} 3 & 2 & 4 & 11 \\ -2 & 1 & 5 & 6 \\ 5 & -7 & -20 & 15 \\ 4 & -1 & 13 & 12 \end{vmatrix} = \begin{vmatrix} 3 & 2 & -6 & 11 \\ -2 & 1 & 0 & 6 \\ 5 & -7 & 15 & 15 \\ 4 & -1 & 8 & 12 \end{vmatrix}$

17. $\begin{vmatrix} 5 & 4 & 2 \\ 4 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = - \begin{vmatrix} 5 & 4 & 2 \\ -4 & 3 & -4 \\ 7 & 6 & 3 \end{vmatrix}$
18. $\begin{vmatrix} 3 & 2 & -2 \\ -1 & 0 & 3 \\ 4 & 2 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & 2 & -2 \\ 4 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$
19. $\begin{vmatrix} 2 & 1 & -1 & 0 & 4 \\ 1 & 0 & 1 & 3 & 2 \\ 3 & 6 & 1 & -3 & 6 \\ 0 & 4 & 0 & 2 & 0 \\ -1 & 8 & 5 & 3 & 2 \end{vmatrix} = 0$
20. $\begin{vmatrix} 4 & 3 & 1 & 9 & 9 \\ 9 & -1 & 2 & 3 & -3 \\ 3 & 4 & 6 & 9 & 12 \\ 5 & 2 & 0 & 6 & 6 \\ 6 & 0 & 3 & 0 & 0 \end{vmatrix} = 0$

Finding a Determinant In Exercises 21–24, use either elementary row or column operations, or cofactor expansion, to find the determinant by hand. Then use a software program or a graphing utility to verify your answer.

21. $\begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & 4 \\ 2 & 0 & 3 \end{vmatrix}$
22. $\begin{vmatrix} -1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix}$
23. $\begin{vmatrix} 5 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & 1 & 2 \\ -1 & 0 & 3 & 1 \end{vmatrix}$
24. $\begin{vmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{vmatrix}$

Finding a Determinant In Exercises 25–36, use elementary row or column operations to find the determinant.

25. $\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix}$
26. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$
27. $\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ -6 & 3 & 3 \end{vmatrix}$
28. $\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$
29. $\begin{vmatrix} 3 & 2 & -3 \\ 7 & 5 & 1 \\ -1 & 2 & 6 \end{vmatrix}$
30. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{vmatrix}$
31. $\begin{vmatrix} 4 & -7 & 9 & 1 \\ 6 & 2 & 7 & 0 \\ 3 & 6 & -3 & 3 \\ 0 & 7 & 4 & -1 \end{vmatrix}$
32. $\begin{vmatrix} 9 & -4 & 2 & 5 \\ 2 & 7 & 6 & -5 \\ 4 & 1 & -2 & 0 \\ 7 & 3 & 4 & 10 \end{vmatrix}$

$$33. \begin{vmatrix} 1 & -2 & 7 & 9 \\ 3 & -4 & 5 & 5 \\ 3 & 6 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{vmatrix}$$

$$34. \begin{vmatrix} 0 & -4 & 9 & 3 \\ 9 & 2 & -2 & 7 \\ -5 & 7 & 0 & 11 \\ -8 & 0 & 0 & 16 \end{vmatrix}$$

$$35. \begin{vmatrix} 1 & -1 & 8 & 4 & 2 \\ 2 & 6 & 0 & -4 & 3 \\ 2 & 0 & 2 & 6 & 2 \\ 0 & 2 & 8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{vmatrix}$$

$$36. \begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$$

True or False? In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

37. (a) Interchanging two rows of a square matrix changes the sign of its determinant.
 (b) Multiplying a column of a square matrix by a nonzero constant results in the determinant being multiplied by the same nonzero constant.
 (c) If two rows of a square matrix are equal, then its determinant is 0.
38. (a) Adding a multiple of one column of a square matrix to another column changes only the sign of the determinant.
 (b) Two matrices are column-equivalent when one matrix can be obtained by performing elementary column operations on the other.
 (c) If one row of a square matrix is a multiple of another row, then the determinant is 0.

Finding the Determinant of an Elementary Matrix In Exercises 39–42, find the determinant of the elementary matrix. (Assume $k \neq 0$.)

$$39. \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$40. \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$41. \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$42. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

43. Proof Prove the property.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} (a_{11} + b_{11}) & a_{12} & a_{13} \\ (a_{21} + b_{21}) & a_{22} & a_{23} \\ (a_{31} + b_{31}) & a_{32} & a_{33} \end{vmatrix}$$

44. Proof Prove the property.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right),$$

$a \neq 0, \quad b \neq 0, \quad c \neq 0$

45. Find each determinant.

$$(a) \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} \quad (b) \begin{vmatrix} \sin \theta & 1 \\ 1 & \sin \theta \end{vmatrix}$$

46. CAPSTONE Evaluate each determinant when $a = 1$, $b = 4$, and $c = -3$.

$$(a) \begin{vmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix} \quad (b) \begin{vmatrix} a & 0 & 1 \\ 0 & c & 0 \\ b & 0 & -16 \end{vmatrix}$$

47. Guided Proof Prove Property 2 of Theorem 3.3: When B is obtained from A by adding a multiple of a row of A to another row of A , $\det(B) = \det(A)$.

Getting Started: To prove that the determinant of B is equal to the determinant of A , you need to show that their respective cofactor expansions are equal.

- Begin by letting B be the matrix obtained by adding c times the j th row of A to the i th row of A .
- Find the determinant of B by expanding in this i th row.
- Distribute and then group the terms containing a coefficient of c and those not containing a coefficient of c .
- Show that the sum of the terms not containing a coefficient of c is the determinant of A , and the sum of the terms containing a coefficient of c is equal to 0.

48. Guided Proof Prove Property 3 of Theorem 3.3: When B is obtained from A by multiplying a row of A by a nonzero constant c , $\det(B) = c \det(A)$.

Getting Started: To prove that the determinant of B is equal to c times the determinant of A , you need to show that the determinant of B is equal to c times the cofactor expansion of the determinant of A .

- Begin by letting B be the matrix obtained by multiplying c times the i th row of A .
- Find the determinant of B by expanding in this i th row.
- Factor out the common factor c .
- Show that the result is c times the determinant of A .

3.3 Properties of Determinants

-  Find the determinant of a matrix product and a scalar multiple of a matrix.
-  Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.
-  Find the determinant of the transpose of a matrix.

MATRIX PRODUCTS AND SCALAR MULTIPLES

In this section, you will learn several important properties of determinants. You will begin by considering the determinant of the product of two matrices.

EXAMPLE 1 The Determinant of a Matrix Product

Find $|A|$, $|B|$, and $|AB|$ for the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}.$$

SOLUTION

$|A|$ and $|B|$ have the values

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7 \quad \text{and} \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11.$$

The matrix product AB is

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}.$$

Finally,

$$|AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77.$$

In Example 1, note that $|AB| = |A||B|$, or $-77 = (-7)(11)$. This is true in general.

REMARK

Theorem 3.5 can be extended to include the product of any finite number of matrices.

That is,

$$\begin{aligned} &|A_1 A_2 A_3 \cdots A_k| \\ &= |A_1| |A_2| |A_3| \cdots |A_k|. \end{aligned}$$

THEOREM 3.5 Determinant of a Matrix Product

If A and B are square matrices of order n , then $\det(AB) = \det(A) \det(B)$.

PROOF

To begin, observe that if E is an elementary matrix, then, by Theorem 3.3, the next three statements are true. If you obtain E from I by interchanging two rows, then $|E| = -1$. If you obtain E by multiplying a row of I by a nonzero constant c , then $|E| = c$. If you obtain E by adding a multiple of one row of I to another row of I , then $|E| = 1$. Additionally, by Theorem 2.12, if E results from performing an elementary row operation on I and the same elementary row operation is performed on B , then the matrix EB results. It follows that $|EB| = |E| |B|$.

This can be generalized to conclude that $|E_k \cdots E_2 E_1 B| = |E_k| \cdots |E_2| |E_1| |B|$, where E_i is an elementary matrix. Now consider the matrix AB . If A is *nonsingular*, then, by Theorem 2.14, it can be written as the product $A = E_k \cdots E_2 E_1$, and

$$\begin{aligned} |AB| &= |E_k \cdots E_2 E_1 B| \\ &= |E_k| \cdots |E_2| |E_1| |B| \\ &= |E_k \cdots E_2 E_1| |B| \\ &= |A| |B|. \end{aligned}$$

If A is *singular*, then A is row-equivalent to a matrix with an entire row of zeros. From Theorem 3.4, $|A| = 0$. Moreover, it follows that AB is also singular. (If AB were nonsingular, then $A[B(AB)^{-1}] = I$ would imply that A is nonsingular.) So, $|AB| = 0$, and you can conclude that $|AB| = |A| |B|$. 

The next theorem shows the relationship between $|A|$ and $|cA|$.

THEOREM 3.6 Determinant of a Scalar Multiple of a Matrix

If A is a square matrix of order n and c is a scalar, then the determinant of cA is

$$\det(cA) = c^n \det(A).$$

PROOF

This formula can be proven by repeated applications of Property 3 of Theorem 3.3. Factor the scalar c out of each of the n rows of $|cA|$ to obtain $|cA| = c^n |A|$. 

EXAMPLE 2

The Determinant of a Scalar Multiple of a Matrix

Find the determinant of the matrix.

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}$$

SOLUTION

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

so apply Theorem 3.6 to conclude that

$$|A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 1000(5) = 5000. \quad \text{img alt="blue square icon" data-bbox="925 735 942 750"/>$$

Theorems 3.5 and 3.6 give formulas for the determinants of the product of two matrices and a scalar multiple of a matrix. These theorems do not, however, give a formula for the determinant of the *sum* of two matrices. The sum of the determinants of two matrices usually does not equal the determinant of their sum. That is, in general, $|A| + |B| \neq |A + B|$. For example, if

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

then $|A| = 2$ and $|B| = -3$, but $A + B = \begin{bmatrix} 9 & 9 \\ 2 & 0 \end{bmatrix}$ and $|A + B| = -18$.

DETERMINANTS AND THE INVERSE OF A MATRIX

It can be difficult to tell simply by inspection whether a matrix has an inverse. Can you tell which of the matrices below is invertible?

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

The next theorem suggests that determinants are useful for classifying square matrices as invertible or noninvertible.

THEOREM 3.7 Determinant of an Invertible Matrix

A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$.

PROOF

To prove the theorem in one direction, assume A is invertible. Then $AA^{-1} = I$, and by Theorem 3.5 you can write $|A||A^{-1}| = |I|$. Now, $|I| = 1$, so you know that neither determinant on the left is zero. Specifically, $|A| \neq 0$.

To prove the theorem in the other direction, assume the determinant of A is nonzero. Then, using Gauss-Jordan elimination, find a matrix B , in reduced row-echelon form, that is row-equivalent to A . The matrix B must be the identity matrix I or it must have at least one row that consists entirely of zeros, because B is in reduced row-echelon form. But if B has a row of all zeros, then by Theorem 3.4 you know that $|B| = 0$, which would imply that $|A| = 0$. You assumed that $|A|$ is nonzero, so you can conclude that $B = I$. The matrix A is, therefore, row-equivalent to the identity matrix, and by Theorem 2.15 you know that A is invertible. 

DISCOVERY

Let

$$A = \begin{bmatrix} 6 & 4 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Use a software program or a graphing utility to find A^{-1} .
2. Compare $\det(A^{-1})$ with $\det(A)$.
3. Make a conjecture about the determinant of the inverse of a matrix.

EXAMPLE 3

Classifying Square Matrices as Singular or Nonsingular

Determine whether each matrix has an inverse.

$$\text{a. } \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad \text{b. } \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

SOLUTION

$$\text{a. } \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0$$

so this matrix has no inverse (it is singular).

$$\text{b. } \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12 \neq 0$$

so this matrix has an inverse (it is nonsingular). 

The next theorem provides a way to find the determinant of an inverse matrix.

THEOREM 3.8 Determinant of an Inverse Matrix

If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

PROOF

The matrix A is invertible, so $AA^{-1} = I$, and using Theorem 3.5, $|A||A^{-1}| = |I| = 1$. By Theorem 3.7, you know that $|A| \neq 0$, so you can divide each side by $|A|$ to obtain

$$|A^{-1}| = \frac{1}{|A|}.$$

EXAMPLE 4**The Determinant of the Inverse of a Matrix**

Find $|A^{-1}|$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

SOLUTION

One way to solve this problem is to find A^{-1} and then evaluate its determinant. It is easier, however, to apply Theorem 3.8, as shown below. Find the determinant of A ,

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

and then use the formula $|A^{-1}| = 1/|A|$ to conclude that $|A^{-1}| = \frac{1}{4}$.

REMARK

The inverse of A is

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Evaluate the determinant of this matrix directly. Then compare your answer with that obtained in Example 4.

REMARK

In Section 3.2, you saw that a square matrix A has a determinant of zero when A is row-equivalent to a matrix that has at least one row consisting entirely of zeros. The validity of this statement follows from the equivalence of Statements 4 and 6.

Equivalent Conditions for a Nonsingular Matrix

If A is an $n \times n$ matrix, then the statements below are equivalent.

1. A is invertible.
2. $Ax = b$ has a unique solution for every $n \times 1$ column matrix b .
3. $Ax = 0$ has only the trivial solution.
4. A is row-equivalent to I_n .
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$

EXAMPLE 5**Systems of Linear Equations**

Which of the systems has a unique solution?

$$\begin{array}{ll} \text{a.} & \begin{array}{l} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = -4 \end{array} \\ \text{b.} & \begin{array}{l} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 + x_3 = -4 \end{array} \end{array}$$

SOLUTION

From Example 3, you know that the coefficient matrices for these two systems have the determinants shown below.

$$\text{a.} \quad \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0 \qquad \text{b.} \quad \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12$$

Using the preceding list of equivalent conditions, you can conclude that only the second system has a unique solution.

DETERMINANTS AND THE TRANSPOSE OF A MATRIX

The next theorem tells you that the determinant of the transpose of a square matrix is equal to the determinant of the original matrix. This theorem can be proven using mathematical induction and Theorem 3.1, which states that a determinant can be evaluated using cofactor expansion in a row or a column. The details of the proof are left to you. (See Exercise 66.)

THEOREM 3.9 Determinant of a Transpose

If A is a square matrix, then

$$\det(A) = \det(A^T).$$

EXAMPLE 6

The Determinant of a Transpose

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Show that $|A| = |A^T|$ for the matrix below.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{bmatrix}$$

SOLUTION

To find the determinant of A , expand by cofactors in the second *row* to obtain

$$\begin{aligned} |A| &= 2(-1)^3 \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$

To find the determinant of

$$A^T = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{bmatrix}$$

expand by cofactors in the second *column* to obtain

$$\begin{aligned} |A^T| &= 2(-1)^3 \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$



LINEAR ALGEBRA APPLIED

Systems of linear differential equations often arise in engineering and control theory. For a function $f(t)$ that is defined for all positive values of t , the **Laplace transform** of $f(t)$ is

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

provided that the improper integral exists. Laplace transforms and Cramer's Rule, which uses determinants to solve a system of linear equations, can sometimes be used to solve a system of differential equations. You will study Cramer's Rule in the next section.

William Perugini/Shutterstock.com

3.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



The Determinant of a Matrix Product In Exercises 1–6, find (a) $|A|$, (b) $|B|$, (c) AB , and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

$$1. A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$$

$$3. A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 2 & 4 & 7 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 6 & 1 & -1 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The Determinant of a Scalar Multiple of a Matrix In Exercises 7–14, use the fact that $|cA| = c^n|A|$ to evaluate the determinant of the $n \times n$ matrix.

$$7. A = \begin{bmatrix} 5 & 15 \\ 10 & -20 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 21 & 7 \\ 28 & -56 \end{bmatrix}$$

$$9. A = \begin{bmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 4 & 16 & 0 \\ 12 & -8 & 8 \\ 16 & 20 & -4 \end{bmatrix}$$

$$11. A = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 6 & -8 \\ 6 & -8 & 10 \end{bmatrix}$$

$$12. A = \begin{bmatrix} 40 & 25 & 10 \\ 30 & 5 & 20 \\ 15 & 35 & 45 \end{bmatrix}$$

$$13. A = \begin{bmatrix} 5 & 0 & -15 & 0 \\ 0 & 5 & 0 & 0 \\ -10 & 0 & 5 & 0 \\ 0 & -20 & 0 & 5 \end{bmatrix}$$

$$14. A = \begin{bmatrix} 0 & 16 & -8 & -32 \\ -16 & 8 & -8 & 16 \\ 8 & -24 & 8 & -8 \\ -8 & 32 & 0 & 32 \end{bmatrix}$$

The Determinant of a Matrix Sum In Exercises 15–18, find (a) $|A|$, (b) $|B|$, (c) $A + B$, and (d) $|A + B|$. Then verify that $|A| + |B| \neq |A + B|$.

$$15. A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$$

$$16. A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$$

$$17. A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$18. A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Classifying Matrices as Singular or Nonsingular In Exercises 19–24, use a determinant to decide whether the matrix is singular or nonsingular.

$$19. \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$$

$$20. \begin{bmatrix} 3 & -6 \\ 4 & 2 \end{bmatrix}$$

$$21. \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$22. \begin{bmatrix} 14 & 5 & 7 \\ -15 & 0 & 3 \\ 1 & -5 & -10 \end{bmatrix}$$

$$23. \begin{bmatrix} 1 & 0 & -8 & 2 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$24. \begin{bmatrix} 0.8 & 0.2 & -0.6 & 0.1 \\ -1.2 & 0.6 & 0.6 & 0 \\ 0.7 & -0.3 & 0.1 & 0 \\ 0.2 & -0.3 & 0.6 & 0 \end{bmatrix}$$

The Determinant of the Inverse of a Matrix In Exercises 25–30, find $|A^{-1}|$. Begin by finding A^{-1} , and then evaluate its determinant. Verify your result by finding $|A|$ and then applying the formula from Theorem 3.8, $|A^{-1}| = \frac{1}{|A|}$.

$$25. A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix}$$

$$27. A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$29. A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 1 & 0 & 3 & -2 \\ 2 & 0 & 2 & -1 \\ 1 & -3 & 1 & 2 \end{bmatrix}$$

$$30. A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 2 & -2 \\ 1 & -2 & -4 & 1 \end{bmatrix}$$

System of Linear Equations In Exercises 31–36, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution.

$$\begin{array}{ll} 31. \begin{cases} x_1 - 3x_2 = 2 \\ 2x_1 + x_2 = 1 \end{cases} & 32. \begin{cases} 3x_1 - 4x_2 = 2 \\ \frac{2}{3}x_1 - \frac{8}{9}x_2 = 1 \end{cases} \end{array}$$

$$\begin{array}{l} 33. \begin{cases} x_1 - x_2 + x_3 = 4 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 2x_3 = 0 \end{cases} \end{array}$$

$$\begin{array}{l} 34. \begin{cases} x_1 + x_2 - x_3 = 4 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 2x_3 = 0 \end{cases} \end{array}$$

$$\begin{array}{l} \text{35. } \begin{cases} 2x_1 + x_2 + 5x_3 + x_4 = 5 \\ x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \\ x_1 + 5x_2 - 6x_3 = 3 \end{cases} \end{array}$$

$$\begin{array}{l} \text{36. } \begin{cases} x_1 - x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \end{cases} \end{array}$$

Singular Matrices In Exercises 37–42, find the value(s) of k such that A is singular.

$$37. A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix} \quad 38. A = \begin{bmatrix} k-1 & 2 \\ 2 & k+2 \end{bmatrix}$$

$$39. A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 4 & 2 & k \end{bmatrix} \quad 40. A = \begin{bmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{bmatrix}$$

$$41. A = \begin{bmatrix} 0 & k & 1 \\ k & 1 & k \\ 1 & k & 0 \end{bmatrix} \quad 42. A = \begin{bmatrix} k & -3 & -k \\ -2 & k & 1 \\ k & 1 & 0 \end{bmatrix}$$

Finding Determinants In Exercises 43–50, find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$.

$$43. A = \begin{bmatrix} 6 & -11 \\ 4 & -5 \end{bmatrix} \quad 44. A = \begin{bmatrix} -4 & 10 \\ 5 & 6 \end{bmatrix}$$

$$45. A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & -1 & 2 \end{bmatrix} \quad 46. A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & -6 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$47. A = \begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & 6 \\ 3 & 2 & 1 \end{bmatrix} \quad 48. A = \begin{bmatrix} 4 & 1 & 9 \\ -1 & 0 & -2 \\ -3 & 3 & 0 \end{bmatrix}$$

$$49. A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$50. A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Finding Determinants In Exercises 51–56, use a software program or a graphing utility to find (a) $|A|$, (b) $|A^T|$, (c) $|A^2|$, (d) $|2A|$, and (e) $|A^{-1}|$.

$$51. A = \begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix} \quad 52. A = \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$53. A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ -3 & 1 & 2 \end{bmatrix}$$

$$54. A = \begin{bmatrix} \frac{3}{4} & \frac{2}{3} & -\frac{1}{4} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{1}{4} & \frac{1}{3} & \frac{3}{4} \end{bmatrix}$$

$$55. A = \begin{bmatrix} 4 & -2 & 1 & 5 \\ 3 & 8 & 2 & -1 \\ 6 & 8 & 9 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}$$

$$56. A = \begin{bmatrix} 6 & 5 & 1 & -1 \\ -2 & 4 & 3 & 5 \\ 6 & 1 & -4 & -2 \\ 2 & 2 & 1 & 3 \end{bmatrix}$$

57. Let A and B be square matrices of order 4 such that $|A| = -5$ and $|B| = 3$. Find (a) $|A^2|$, (b) $|B^2|$, (c) $|A^3|$, and (d) $|B^4|$.

58. CAPSTONE Let A and B be square matrices of order 3 such that $|A| = 4$ and $|B| = 5$.

- (a) Find $|AB|$. (b) Find $|2A|$.
- (c) Are A and B singular or nonsingular? Explain.
- (d) If A and B are nonsingular, find $|A^{-1}|$ and $|B^{-1}|$.
- (e) Find $|(AB)^T|$.

59. Proof Let A and B be $n \times n$ matrices such that $AB = I$. Prove that $|A| \neq 0$ and $|B| \neq 0$.

60. Proof Let A and B be $n \times n$ matrices such that AB is singular. Prove that either A or B is singular.

61. Find two 2×2 matrices such that $|A| + |B| = |A + B|$.

62. Verify the equation.

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$$

63. Let A be an $n \times n$ matrix in which the entries of each row sum to zero. Find $|A|$.

64. Illustrate the result of Exercise 63 with the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

- 65. Guided Proof** Prove that the determinant of an invertible matrix A is equal to ± 1 when all of the entries of A and A^{-1} are integers.

Getting Started: Denote $\det(A)$ as x and $\det(A^{-1})$ as y . Note that x and y are real numbers. To prove that $\det(A)$ is equal to ± 1 , you must show that both x and y are integers such that their product xy is equal to 1.

- Use the property for the determinant of a matrix product to show that $xy = 1$.
- Use the definition of a determinant and the fact that the entries of A and A^{-1} are integers to show that both $x = \det(A)$ and $y = \det(A^{-1})$ are integers.
- Conclude that $x = \det(A)$ must be either 1 or -1 because these are the only integer solutions to the equation $xy = 1$.

- 66. Guided Proof** Prove Theorem 3.9: If A is a square matrix, then $\det(A) = \det(A^T)$.

Getting Started: To prove that the determinants of A and A^T are equal, you need to show that their cofactor expansions are equal. The cofactors are $\pm$ determinants of smaller matrices, so you need to use mathematical induction.

- Initial step for induction: If A is of order 1, then $A = [a_{11}] = A^T$
so
 $\det(A) = \det(A^T) = a_{11}$.
- Assume the inductive hypothesis holds for all matrices of order $n - 1$. Let A be a square matrix of order n . Write an expression for the determinant of A by expanding in the first row.
- Write an expression for the determinant of A^T by expanding in the first column.
- Compare the expansions in (ii) and (iii). The entries of the first row of A are the same as the entries of the first column of A^T . Compare cofactors (these are the $\pm$ determinants of smaller matrices that are transposes of one another) and use the inductive hypothesis to conclude that they are equal as well.

- 67. Writing** Let A and P be $n \times n$ matrices, where P is invertible. Does $P^{-1}AP = A$? Illustrate your conclusion with appropriate examples. What can you say about the two determinants $|P^{-1}AP|$ and $|A|$?

- 68. Writing** Let A be an $n \times n$ nonzero matrix satisfying $A^{10} = O$. Explain why A must be singular. What properties of determinants are you using in your argument?

- 69. Proof** A square matrix is **skew-symmetric** when $A^T = -A$. Prove that if A is an $n \times n$ skew-symmetric matrix, then $|A| = (-1)^n |A|$.

- 70. Proof** Let A be a skew-symmetric matrix of odd order. Use the result of Exercise 69 to prove that $|A| = 0$.

True or False? In Exercises 71 and 72, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows that the statement is not true in all cases or cite an appropriate statement from the text.

- (a) If A is an $n \times n$ matrix and c is a nonzero scalar, then the determinant of the matrix cA is $nc \cdot \det(A)$.
- (b) If A is an invertible matrix, then the determinant of A^{-1} is equal to the reciprocal of the determinant of A .
- (c) If A is an invertible $n \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- (a) The determinant of the sum of two matrices equals the sum of the determinants of the matrices.
- (b) If A and B are square matrices of order n , and $\det(A) = \det(B)$, then $\det(AB) = \det(A^2)$.
- (c) If the determinant of an $n \times n$ matrix A is nonzero, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Orthogonal Matrices In Exercises 73–78, determine whether the matrix is orthogonal. An invertible square matrix A is orthogonal when $A^{-1} = A^T$.

$$73. \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$74. \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$75. \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$76. \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$77. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$78. \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

- 79. Proof** Prove that the $n \times n$ identity matrix is orthogonal.

- 80. Proof** Prove that if A is an orthogonal matrix, then $|A| = \pm 1$.



Orthogonal Matrices In Exercises 81 and 82, use a graphing utility to determine whether A is orthogonal. Then verify that $|A| = \pm 1$.

$$81. A = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

$$82. A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

- 83. Proof** If A is an idempotent matrix ($A^2 = A$), then prove that the determinant of A is either 0 or 1.

- 84. Proof** Let S be an $n \times n$ singular matrix. Prove that for any $n \times n$ matrix B , the matrix SB is also singular.

3.4 Applications of Determinants

-  Find the adjoint of a matrix and use it to find the inverse of the matrix.
-  Use Cramer's Rule to solve a system of n linear equations in n variables.
-  Use determinants to find area, volume, and the equations of lines and planes.

THE ADJOINT OF A MATRIX

So far in this chapter, you have studied procedures for evaluating, and properties of, determinants. In this section, you will study an explicit formula for the inverse of a nonsingular matrix and use this formula to prove a theorem known as Cramer's Rule. You will also use Cramer's Rule to solve systems of linear equations, and study several other applications of determinants.

Recall from Section 3.1 that the cofactor C_{ij} of a square matrix A is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the i th row and j th column of A . The **matrix of cofactors** of A has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is the **adjoint** of A and is denoted $\text{adj}(A)$. That is,

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

EXAMPLE 1

Finding the Adjoint of a Square Matrix

Find the adjoint of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

SOLUTION

The cofactor C_{11} is

$$\begin{bmatrix} \boxed{-1} & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \rightarrow C_{11} = (-1)^2 \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4.$$

Continuing this process produces the matrix of cofactors of A shown below.

$$\begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

The transpose of this matrix is the adjoint of A . That is, $\text{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$. 

REMARK

Theorem 3.10 is not particularly efficient for finding the inverse of a matrix. The Gauss-Jordan elimination method discussed in Section 2.3 is much more efficient. Theorem 3.10 is theoretically useful, however, because it provides a concise formula for the inverse of a matrix.

REMARK

If A is a 2×2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the adjoint of A is simply

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Moreover, if A is invertible, then from Theorem 3.10 you have

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} \text{adj}(A) \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

which agrees with the formula given in Section 2.3.

The adjoint of a matrix A can be useful for finding the inverse of A , as shown in the next theorem.

THEOREM 3.10 The Inverse of a Matrix Using Its Adjoint

If A is an $n \times n$ invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

PROOF

Begin by proving that the product of A and its adjoint is equal to the product of the determinant of A and I_n . Consider the product

$$A[\text{adj}(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the i th row and j th column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

If $i = j$, then this sum is simply the cofactor expansion of A in its i th row, which means that the sum is the determinant of A . On the other hand, if $i \neq j$, then the sum is zero. (Verify this.)

$$A[\text{adj}(A)] = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

The matrix A is invertible, so $\det(A) \neq 0$ and you can write

$$\frac{1}{\det(A)} A[\text{adj}(A)] = I \quad \text{or} \quad A \left[\frac{1}{\det(A)} \text{adj}(A) \right] = I.$$

By Theorem 2.7 and the definition of the inverse of a matrix, it follows that

$$\frac{1}{\det(A)} \text{adj}(A) = A^{-1}.$$

EXAMPLE 2**Using the Adjoint of a Matrix to Find Its Inverse**

Use the adjoint of A to find A^{-1} , where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

SOLUTION

The determinant of this matrix is 3. Using the adjoint of A (found in Example 1), the inverse of A is

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$

Check that this matrix is the inverse of A by showing that $AA^{-1} = I = A^{-1}A$.

CRAMER'S RULE

Cramer's Rule, named after Gabriel Cramer (1704–1752), uses determinants to solve a system of n linear equations in n variables. This rule applies only to systems with unique solutions. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 3.1. There, it was pointed out that the system

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$. A determinant can represent each numerator and denominator in this solution, as shown below.

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

The denominator for x_1 and x_2 is simply the determinant of the coefficient matrix A of the original system. The numerators for x_1 and x_2 are formed by using the column of constants as replacements for the coefficients of x_1 and x_2 in $|A|$. These two determinants are denoted by $|A_1|$ and $|A_2|$, as shown below.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{and} \quad |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

You have $x_1 = \frac{|A_1|}{|A|}$ and $x_2 = \frac{|A_2|}{|A|}$. This determinant form of the solution is called **Cramer's Rule**.

EXAMPLE 3

Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations.

$$4x_1 - 2x_2 = 10$$

$$3x_1 - 5x_2 = 11$$

SOLUTION

First find the determinant of the coefficient matrix.

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -14$$

The determinant is nonzero, so you know the system has a unique solution, and applying Cramer's Rule produces

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-28}{-14} = 2$$

and

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{14}{-14} = -1.$$

The solution is $x_1 = 2$ and $x_2 = -1$.



Cramer's Rule generalizes to systems of n linear equations in n variables. The value of each variable is the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For example, x_3 in the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \text{is} \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

THEOREM 3.11 Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the i th column of A_i is the column of constants in the system of equations.

PROOF

Let the system be represented by $AX = B$. The determinant of A is nonzero, so you can write

$$X = A^{-1}B = \frac{1}{|A|}\text{adj}(A)B = [x_1 \ x_2 \ \dots \ x_n]^T.$$

If the entries of B are $b_1, b_2, \dots, b_n$, then $x_1 = \frac{1}{|A|}(b_1C_{11} + b_2C_{21} + \dots + b_nC_{n1})$, but the sum (in parentheses) is precisely the cofactor expansion of A_{11} , which means that $x_1 = |A_{11}|/|A|$, and the proof is complete. 

EXAMPLE 4

Using Cramer's Rule

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Use Cramer's Rule to solve the system of linear equations for x .

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x &+ z = 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

SOLUTION

The determinant of the coefficient matrix is $|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10$.

The determinant is nonzero, so you know that the solution is unique. Apply Cramer's Rule to solve for x , as shown below.

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

REMARK

Apply Cramer's Rule to solve for y and z . You will see that the solution is $y = -\frac{3}{2}$ and $z = -\frac{8}{5}$.

AREA, VOLUME, AND EQUATIONS OF LINES AND PLANES

Determinants have many applications in analytic geometry. One application is in finding the area of a triangle in the xy -plane.

Area of a Triangle in the xy -Plane

The area of a triangle with vertices

$$(x_1, y_1), (x_2, y_2), \text{ and } (x_3, y_3)$$

is

$$\text{Area} = \pm \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

PROOF

Prove the case for $y_i > 0$. Assume that $x_1 \leq x_3 \leq x_2$ and that (x_3, y_3) lies above the line segment connecting (x_1, y_1) and (x_2, y_2) , as shown in Figure 3.1. Consider the three trapezoids whose vertices are

Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$

Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$

Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$.

The area of the triangle is equal to the sum of the areas of the first two trapezoids minus the area of the third trapezoid. So,

$$\begin{aligned} \text{Area} &= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1) \\ &= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2) \\ &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \end{aligned}$$

If the vertices do not occur in the order $x_1 \leq x_3 \leq x_2$ or if the vertex (x_3, y_3) is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use $\pm$ and choose the correct sign to give a positive area. ■

EXAMPLE 5

Finding the Area of a Triangle

Find the area of the triangle whose vertices are

$$(1, 1), (2, 2), \text{ and } (4, 3).$$

SOLUTION

It is not necessary to know the relative positions of the three vertices. Simply evaluate the determinant

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{1}{2}$$

and conclude that the area of the triangle is $\frac{1}{2}$ square unit. ■

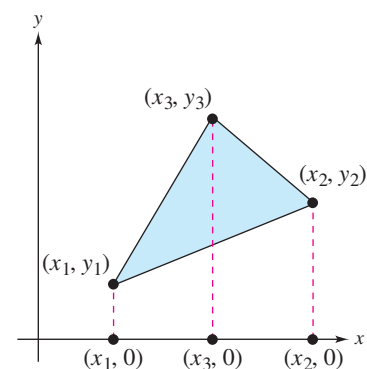


Figure 3.1

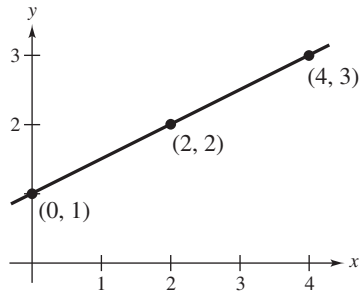


Figure 3.2

If the three points in Example 5 had been on the same line, what would have happened when you applied the area formula? The answer is that the determinant would have been zero. Consider, for example, the three collinear points $(0, 1)$, $(2, 2)$, and $(4, 3)$, as shown in Figure 3.2. The determinant that yields the area of the “triangle” that has these three points as vertices is

$$\frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0.$$

If three points in the xy -plane lie on the same line, then the determinant in the formula for the area of a triangle is zero, as generalized below.

Test for Collinear Points in the xy -Plane

Three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are collinear if and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

The test for collinear points can be adapted to another use. That is, when you are given two points in the xy -plane, you can find an equation of the line passing through the two points, as shown below.

Two-Point Form of an Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

EXAMPLE 6

Finding an Equation of the Line Passing Through Two Points

Find an equation of the line passing through the points

$$(2, 4) \quad \text{and} \quad (-1, 3).$$

SOLUTION

Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for an equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, expand by cofactors in the first row.

$$\begin{aligned} x \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} &= 0 \\ x(1) - y(3) + 1(10) &= 0 \\ x - 3y + 10 &= 0 \end{aligned}$$

So, an equation of the line is $x - 3y = -10$.

The formula for the area of a triangle in the plane has a straightforward generalization to three-dimensional space, which is presented below without proof.

Volume of a Tetrahedron

The volume of a tetrahedron with vertices (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and (x_4, y_4, z_4) is

$$\text{Volume} = \pm \frac{1}{6} \det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive volume.

EXAMPLE 7

Finding the Volume of a Tetrahedron

Find the volume of the tetrahedron shown in Figure 3.3.

SOLUTION

Using the determinant formula for the volume of a tetrahedron produces

$$\frac{1}{6} \begin{vmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix} = \frac{1}{6}(-72) = -12.$$

So, the volume of the tetrahedron is 12 cubic units.

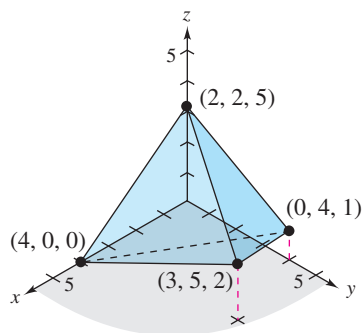


Figure 3.3

If four points in three-dimensional space lie in the same plane, then the determinant in the formula for the volume of a tetrahedron is zero. So, you have the test shown below.

Test for Coplanar Points in Space

Four points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and (x_4, y_4, z_4) are coplanar if and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

An adaptation of this test is the determinant form of an equation of a plane passing through three points in space, as shown below.

Three-Point Form of an Equation of a Plane

An equation of the plane passing through the distinct points (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) is

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0.$$

EXAMPLE 8**Finding an Equation of the Plane Passing Through Three Points**

Find an equation of the plane passing through the points $(0, 1, 0)$, $(-1, 3, 2)$, and $(-2, 0, 1)$.

SOLUTION

Using the determinant form of an equation of a plane produces

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, subtract the fourth column from the second column to obtain

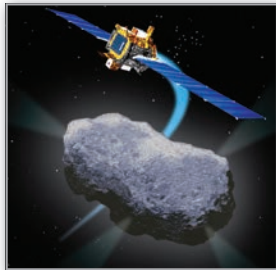
$$\begin{vmatrix} x & y-1 & z & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 1 \\ -2 & -1 & 1 & 1 \end{vmatrix} = 0.$$

Expand by cofactors in the second row.

$$x \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - (y-1) \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} + z \begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$x(4) - (y-1)(3) + z(5) = 0$$

This produces the equation $4x - 3y + 5z = -3$.

**LINEAR ALGEBRA APPLIED**

On November 12, 2014, European Space Agency's Rosetta orbiting spacecraft landed the probe Philae on the surface of the comet 67P/Churyumov-Gerasimenko. Comets that orbit the Sun, such as 67P, follow Kepler's First Law of Planetary Motion. This law states that the orbit is an ellipse, with the sun at one focus of the ellipse. The general equation of a conic section, such as an ellipse, is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To determine the equation of the comet's orbit, astronomers can find the coordinates of the comet at five different points (x_i, y_i) , where $i = 1, 2, 3, 4$, and 5, substitute these coordinates into the equation

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

and then expand by cofactors in the first row to find a , b , c , d , e , and f . For example, the coefficient of x^2 is

$$a = \begin{vmatrix} x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}.$$

Knowing the equation of 67P's orbit helped astronomers determine the ideal time to release the probe.

Jet Propulsion Laboratory/NASA

3.4 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding the Adjoint and Inverse of a Matrix In Exercises 1–8, find the adjoint of the matrix A . Then use the adjoint to find the inverse of A (if possible).

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -2 \end{bmatrix}$

7. $A = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 3 & -1 & 4 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Using Cramer's Rule In Exercises 9–22, use Cramer's Rule to solve (if possible) the system of linear equations.

9. $x_1 + 2x_2 = 5$
 $-x_1 + x_2 = 1$

10. $2x - y = -10$
 $3x + 2y = -1$

11. $3x + 4y = -2$
 $5x + 3y = 4$

12. $18x_1 + 12x_2 = 13$
 $30x_1 + 24x_2 = 23$

13. $20x + 8y = 11$
 $12x - 24y = 21$

14. $13x - 6y = 17$
 $26x - 12y = 8$

15. $-0.4x_1 + 0.8x_2 = 1.6$
 $2x_1 - 4x_2 = 5.0$

16. $-0.4x_1 + 0.8x_2 = 1.6$
 $0.2x_1 + 0.3x_2 = 0.6$

17. $4x - y - z = 1$
 $2x + 2y + 3z = 10$
 $5x - 2y - 2z = -1$

18. $4x - 2y + 3z = -2$
 $2x + 2y + 5z = 16$
 $8x - 5y - 2z = 4$

19. $3x + 4y + 4z = 11$
 $4x - 4y + 6z = 11$
 $6x - 6y = 3$

20. $14x_1 - 21x_2 - 7x_3 = -21$
 $-4x_1 + 2x_2 - 2x_3 = 2$
 $56x_1 - 21x_2 + 7x_3 = 7$

21. $4x_1 - x_2 + x_3 = -5$
 $2x_1 + 2x_2 + 3x_3 = 10$
 $5x_1 - 2x_2 + 6x_3 = 1$

22. $2x_1 + 3x_2 + 5x_3 = 4$
 $3x_1 + 5x_2 + 9x_3 = 7$
 $5x_1 + 9x_2 + 17x_3 = 13$



Using Cramer's Rule In Exercises 23–26, use a software program or a graphing utility and Cramer's Rule to solve (if possible) the system of linear equations.

23. $\frac{5}{6}x_1 - x_2 = -20$
 $\frac{4}{3}x_1 - \frac{7}{2}x_2 = -51$

24. $-8x_1 + 7x_2 - 10x_3 = -151$
 $12x_1 + 3x_2 - 5x_3 = 86$
 $15x_1 - 9x_2 + 2x_3 = 187$

25. $3x_1 - 2x_2 + 9x_3 + 4x_4 = 35$
 $-x_1 - 9x_3 - 6x_4 = -17$
 $3x_3 + x_4 = 5$
 $2x_1 + 2x_2 + 8x_4 = -4$

26. $-x_1 - x_2 + x_4 = -8$
 $3x_1 + 5x_2 + 5x_3 = 24$
 $2x_3 + x_4 = -6$
 $-2x_1 - 3x_2 - 3x_3 = -15$

27. Use Cramer's Rule to solve the system of linear equations for x and y .

$$kx + (1 - k)y = 1$$
$$(1 - k)x + ky = 3$$

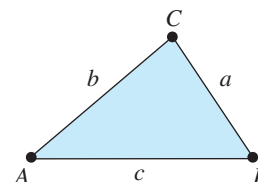
For what value(s) of k will the system be inconsistent?

28. Verify the system of linear equations in $\cos A$, $\cos B$, and $\cos C$ for the triangle shown.

$$c \cos B + b \cos C = a$$
$$c \cos A + a \cos C = b$$
$$b \cos A + a \cos B = c$$

Then use Cramer's Rule to solve for $\cos C$, and use the result to verify the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$



Finding the Area of a Triangle In Exercises 29–32, find the area of the triangle with the given vertices.

29. $(0, 0), (2, 0), (0, 3)$

30. $(1, 1), (2, 4), (4, 2)$

31. $(-1, 2), (2, 2), (-2, 4)$

32. $(1, 1), (-1, 1), (0, -2)$

Testing for Collinear Points In Exercises 33–36, determine whether the points are collinear.

33. $(1, 2), (3, 4), (5, 6)$

34. $(-1, 0), (1, 1), (3, 3)$

35. $(-2, 5), (0, -1), (3, -9)$

36. $(-1, -3), (-4, 7), (2, -13)$

Finding an Equation of a Line In Exercises 37–40, find an equation of the line passing through the points.

37. (0, 0), (3, 4) 38. (−4, 7), (2, 4)
 39. (−2, 3), (−2, −4) 40. (1, 4), (3, 4)

Finding the Volume of a Tetrahedron In Exercises 41–46, find the volume of the tetrahedron with the given vertices.

41. (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)
 42. (1, 1, 1), (0, 0, 0), (2, 1, −1), (−1, 1, 2)
 43. (3, −1, 1), (4, −4, 4), (1, 1, 1), (0, 0, 1)
 44. (0, 0, 0), (0, 2, 0), (3, 0, 0), (1, 1, 4)
 45. (−3, −3, −3), (3, −1, −3), (−3, −1, −3), (−2, 3, 2)
 46. (5, 4, −3), (4, −6, −4), (−6, −6, −5), (0, 0, 10)

Testing for Coplanar Points In Exercises 47–52, determine whether the points are coplanar.

47. (−4, 1, 0), (0, 1, 2), (4, 3, −1), (0, 0, 1)
 48. (1, 2, 3), (−1, 0, 1), (0, −2, −5), (2, 6, 11)
 49. (0, 0, −1), (0, −1, 0), (1, 1, 0), (2, 1, 2)
 50. (1, 2, 7), (−3, 6, 6), (4, 4, 2), (3, 3, 4)
 51. (−3, −2, −1), (2, −1, −2), (−3, −1, −2), (3, 2, 1)
 52. (1, −5, 9), (−1, −5, 9), (1, −5, −9), (−1, −5, −9)

Finding an Equation of a Plane In Exercises 53–58, find an equation of the plane passing through the points.

53. (1, −2, 1), (−1, −1, 7), (2, −1, 3)
 54. (0, −1, 0), (1, 1, 0), (2, 1, 2)
 55. (0, 0, 0), (1, −1, 0), (0, 1, −1)
 56. (1, 2, 7), (4, 4, 2), (3, 3, 4)
 57. (−4, −4, −4), (4, −1, −4), (−4, −1, −4)
 58. (3, 2, −2), (3, −2, 2), (−3, −2, −2)

Using Cramer's Rule In Exercises 59 and 60, determine whether Cramer's Rule is used correctly to solve for the variable. If not, identify the mistake.

$$\begin{array}{lcl}
 \begin{array}{l} x + 2y + z = 2 \\ 59. \quad -x + 3y - 2z = 4 \\ \quad \quad 4x + y - z = 6 \end{array} & y = & \begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & -2 \\ 4 & 1 & -1 \end{vmatrix} \\
 \begin{array}{l} 5x - 2y + z = 15 \\ 60. \quad 3x - 3y - z = -7 \\ \quad \quad 2x - y - 7z = -3 \end{array} & x = & \begin{vmatrix} 15 & -2 & 1 \\ -7 & -3 & -1 \\ -3 & -1 & -7 \end{vmatrix}
 \end{array}$$

61. **Software Publishing** The table shows the estimated revenues (in billions of dollars) of software publishers in the United States from 2011 through 2013. (Source: U.S. Census Bureau)

Year	Revenues, y
2011	156.8
2012	161.7
2013	177.2

- (a) Create a system of linear equations for the data to fit the curve
 $y = at^2 + bt + c$
 where $t = 1$ corresponds to 2011, and y is the revenue.
 (b) Use Cramer's Rule to solve the system.
 (c) Use a graphing utility to plot the data and graph the polynomial function in the same viewing window.
 (d) Briefly describe how well the polynomial function fits the data.

62. **CAPSTONE** Consider the system of linear equations

$$\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}$$

where a_1 , b_1 , c_1 , a_2 , b_2 , and c_2 represent real numbers. What must be true about the lines represented by the equations when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0?$$

63. **Proof** Prove that if $|A| = 1$ and all entries of A are integers, then all entries of $|A^{-1}|$ must also be integers.
 64. **Proof** Prove that if an $n \times n$ matrix A is not invertible, then $A[\text{adj}(A)]$ is the zero matrix.

Proof In Exercises 65 and 66, prove the formula for a nonsingular $n \times n$ matrix A . Assume $n \geq 2$.

65. $|\text{adj}(A)| = |A|^{n-1}$ 66. $\text{adj}[\text{adj}(A)] = |A|^{n-2}A$
 67. Illustrate the formula in Exercise 65 using a nonsingular 2×2 matrix A .
 68. Illustrate the formula in Exercise 66 using a nonsingular 2×2 matrix A .
 69. **Proof** Prove that if A is an $n \times n$ invertible matrix, then $\text{adj}(A^{-1}) = [\text{adj}(A)]^{-1}$.
 70. Illustrate the formula in Exercise 69 using a nonsingular 2×2 matrix A .

3 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



The Determinant of a Matrix In Exercises 1–18, find the determinant of the matrix.

1. $\begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 3 & -4 \\ 0 & -2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

8. $\begin{bmatrix} -15 & 0 & 4 \\ 3 & 0 & -5 \\ 12 & 0 & 6 \end{bmatrix}$

9. $\begin{bmatrix} -3 & 6 & 9 \\ 9 & 12 & -3 \\ 0 & 15 & -6 \end{bmatrix}$

10. $\begin{bmatrix} -15 & 0 & 3 \\ 3 & 9 & -6 \\ 12 & -3 & 6 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ -2 & 0 & 3 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 5 & 2 & 1 & -1 \end{bmatrix}$

13. $\begin{bmatrix} -4 & 1 & 2 & 3 \\ 1 & -2 & 1 & 2 \\ 2 & -1 & 3 & 4 \\ 1 & 2 & 2 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -1 & 2 & 1 \\ -2 & 0 & 1 & -3 \\ -1 & 2 & -3 & 4 \\ -2 & 1 & -2 & 1 \end{bmatrix}$



15. $\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}$



16. $\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 3 & -1 & 2 & -2 \\ 1 & 2 & 0 & 1 & -1 \\ 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix}$

17. $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$

Properties of Determinants In Exercises 19–22, determine which property of determinants the equation illustrates.

19. $\begin{vmatrix} 4 & -1 \\ 16 & -4 \end{vmatrix} = 0$

20. $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 4 & -1 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 4 & 1 & -1 \end{vmatrix}$

21. $\begin{vmatrix} 2 & -4 & 3 & 2 \\ 0 & 4 & 6 & 1 \\ 1 & 8 & 9 & 0 \\ 6 & 12 & -6 & 1 \end{vmatrix} = -12 \begin{vmatrix} 2 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 1 & -2 & 3 & 0 \\ 6 & -3 & -2 & 1 \end{vmatrix}$

22. $\begin{vmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & 2 & 1 \end{vmatrix}$

The Determinant of a Matrix Product In Exercises 23 and 24, find (a) $|A|$, (b) $|B|$, (c) $|AB|$, and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

23. $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 7 & 6 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 3 & -2 \end{bmatrix}$

Finding Determinants In Exercises 25 and 26, find (a) $|A^T|$, (b) $|A^3|$, (c) $|A^T A|$, and (d) $|5A|$.

25. $A = \begin{bmatrix} -3 & 8 \\ 4 & 1 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

Finding Determinants In Exercises 27 and 28, find (a) $|A|$ and (b) $|A^{-1}|$.

27. $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 2 \\ -2 & 7 & 6 \end{bmatrix}$

28. $A = \begin{bmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ -1 & 5 & 0 \end{bmatrix}$

The Determinant of the Inverse of a Matrix In Exercises 29–32, find $|A^{-1}|$. Begin by finding A^{-1} , and then evaluate its determinant. Verify your result by finding $|A|$ and then applying the formula from Theorem 3.8, $|A^{-1}| = \frac{1}{|A|}$.

29. $A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$

30. $A = \begin{bmatrix} 10 & 2 \\ -2 & 7 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 4 \\ 2 & 6 & 0 \end{bmatrix}$

32. $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & 8 \\ 1 & -1 & 0 \end{bmatrix}$

Solving a System of Linear Equations In Exercises 33–36, solve the system of linear equations by each of the methods listed below.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer's Rule

33. $3x_1 + 3x_2 + 5x_3 = 1$

$3x_1 + 5x_2 + 9x_3 = 2$

$5x_1 + 9x_2 + 17x_3 = 4$

34. $x_1 + 2x_2 + x_3 = 4$

$-3x_1 + x_2 - 2x_3 = 1$

$2x_1 + 3x_2 - x_3 = 9$

35. $x_1 + 2x_2 - x_3 = -7$

$2x_1 - 2x_2 - 2x_3 = -8$

$-x_1 + 3x_2 + 4x_3 = 8$

36. $2x_1 + 3x_2 + 5x_3 = 4$

$3x_1 + 5x_2 + 9x_3 = 7$

$5x_1 + 9x_2 + 13x_3 = 17$

System of Linear Equations In Exercises 37–42, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution.

37. $6x + 5y = 0$
 $x - y = 22$

38. $2x - 5y = 2$
 $3x - 7y = 1$

39. $-x + y + 2z = 1$
 $2x + 3y + z = -2$
 $5x + 4y + 2z = 4$

40. $2x + 3y + z = 10$
 $2x - 3y - 3z = 22$
 $8x + 6y = -2$

41. $x_1 + 2x_2 + 6x_3 = 1$
 $2x_1 + 5x_2 + 15x_3 = 4$
 $3x_1 + x_2 + 3x_3 = -6$



42. $x_1 + 5x_2 + 3x_3 = 14$
 $4x_1 + 2x_2 + 5x_3 = 3$
 $3x_3 + 8x_4 + 6x_5 = 16$
 $2x_1 + 4x_2 - 2x_5 = 0$
 $2x_1 - x_3 = 0$

43. Let A and B be square matrices of order 4 such that $|A| = 4$ and $|B| = 2$. Find (a) $|BA|$, (b) $|B^2|$, (c) $|2A|$, (d) $|(AB)^T|$, and (e) $|B^{-1}|$.

44. Let A and B be square matrices of order 3 such that $|A| = -2$ and $|B| = 5$. Find (a) $|BA|$, (b) $|B^4|$, (c) $|2A|$, (d) $|(AB)^T|$, and (e) $|B^{-1}|$.

45. **Proof** Prove the property below.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_{31} & a_{32} + c_{32} & a_{33} + c_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

46. Illustrate the property in Exercise 45 with A , c_{31} , c_{32} , and c_{33} below.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \quad c_{31} = 3, \quad c_{32} = 0, \quad c_{33} = 1$$

47. Find the determinant of the $n \times n$ matrix.

$$\begin{bmatrix} 1-n & 1 & 1 & \dots & 1 \\ & 1 & 1-n & 1 & \dots & 1 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 1 & 1 & 1 & \dots & 1-n \end{bmatrix}$$

48. Show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3.$$

Calculus In Exercises 49–54, find the Jacobians of the functions. If x , y , and z are continuous functions of u , v , and w with continuous first partial derivatives, then the Jacobians $J(u, v)$ and $J(u, v, w)$ are

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{and} \quad J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

49. $x = \frac{1}{2}(v - u)$, $y = \frac{1}{2}(v + u)$

50. $x = au + bv$, $y = cu + dv$

51. $x = u \cos v$, $y = u \sin v$

52. $x = e^u \sin v$, $y = e^u \cos v$

53. $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$, $z = 2uvw$

54. $x = u - v + w$, $y = 2uv$, $z = u + v + w$

55. **Writing** Compare the various methods for calculating the determinant of a matrix. Which method requires the least amount of computation? Which method do you prefer when the matrix has very few zeros?

56. **Writing** Use the table on page 122 to compare the numbers of operations involved in calculating the determinant of a 10×10 matrix by cofactor expansion and then by row reduction. Which method would you prefer to use for calculating determinants?

57. **Writing** Solve the equation for x , if possible.

$$\begin{vmatrix} \cos x & 0 & \sin x \\ \sin x & 0 & \cos x \\ \sin x - \cos x & 1 & \sin x + \cos x \end{vmatrix} = 0$$

58. **Proof** Prove that if $|A| = |B| \neq 0$, and A and B are of the same size, then there exists a matrix C such that

$$|C| = 1 \quad \text{and} \quad A = CB.$$

Finding the Adjoint of a Matrix In Exercises 59 and 60, find the adjoint of the matrix.

59. $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ 60. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

System of Linear Equations In Exercises 61–64, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution. If it does, use Cramer's Rule to find the solution.

61. $0.2x - 0.1y = 0.07$ 62. $2x + y = 0.3$
 $0.4x - 0.5y = -0.01$ $3x - y = -1.3$

63. $2x_1 + 3x_2 + 3x_3 = 3$
 $6x_1 + 6x_2 + 12x_3 = 13$
 $12x_1 + 9x_2 - x_3 = 2$

64. $4x_1 + 4x_2 + 4x_3 = 5$
 $4x_1 - 2x_2 - 8x_3 = 1$
 $8x_1 + 2x_2 - 4x_3 = 6$

 **Using Cramer's Rule** In Exercises 65 and 66, use a software program or a graphing utility and Cramer's Rule to solve (if possible) the system of linear equations.

65. $0.2x_1 - 0.6x_2 = 2.4$
 $-x_1 + 1.4x_2 = -8.8$

66. $4x_1 - x_2 + x_3 = -5$
 $2x_1 + 2x_2 + 3x_3 = 10$
 $5x_1 - 2x_2 + 6x_3 = 1$

Finding the Area of a Triangle In Exercises 67 and 68, use a determinant to find the area of the triangle with the given vertices.

67. $(1, 0), (5, 0), (5, 8)$ 68. $(-4, 0), (4, 0), (0, 6)$

Finding an Equation of a Line In Exercises 69 and 70, use a determinant to find an equation of the line passing through the points.

69. $(-4, 0), (4, 4)$ 70. $(2, 5), (6, -1)$

Finding an Equation of a Plane In Exercises 71 and 72, use a determinant to find an equation of the plane passing through the points.

71. $(0, 0, 0), (1, 0, 3), (0, 3, 4)$

72. $(0, 0, 0), (2, -1, 1), (-3, 2, 5)$

73. Using Cramer's Rule Determine whether Cramer's Rule is used correctly to solve for the variable. If not, identify the mistake.

$$\begin{array}{rcl} x - 4y - z & = & -1 \\ 2x - 3y + z & = & 6 \\ x + y - 4z & = & 1 \end{array} \quad z = \frac{\begin{vmatrix} -1 & -4 & -1 \\ 6 & -3 & 1 \\ 1 & 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & -1 \\ 2 & -3 & 1 \\ 1 & 1 & -4 \end{vmatrix}}$$

74. Health Care Expenditures The table shows annual personal health care expenditures (in billions of dollars) in the United States from 2011 through 2013. (Source: Bureau of Economic Analysis)

Year	2011	2012	2013
Amount, y	1765	1855	1920

(a) Create a system of linear equations for the data to fit the curve

$$y = at^2 + bt + c$$

where $t = 1$ corresponds to 2011, and y is the amount of the expenditure.

(b) Use Cramer's Rule to solve the system.



(c) Use a graphing utility to plot the data and graph the polynomial function in the same viewing window.

(d) Briefly describe how well the polynomial function fits the data.

True or False? In Exercises 75–78, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

75. (a) The cofactor C_{22} of a matrix is always a positive number.

(b) If a square matrix B is obtained from A by interchanging two rows, then $\det(B) = \det(A)$.

(c) If one column of a square matrix is a multiple of another column, then the determinant is 0.

(d) If A is a square matrix of order n , then $\det(A) = -\det(A^T)$.

76. (a) If A and B are square matrices of order n such that $\det(AB) = -1$, then both A and B are nonsingular.

(b) If A is a 3×3 matrix with $\det(A) = 5$, then $\det(2A) = 10$.

(c) If A and B are square matrices of order n , then $\det(A + B) = \det(A) + \det(B)$.

77. (a) In Cramer's Rule, the value of x_i is the quotient of two determinants, where the numerator is the determinant of the coefficient matrix.

(b) Three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are collinear when the determinant of the matrix that has the coordinates as entries in the first two columns and 1's as entries in the third column is nonzero.

78. (a) The matrix of cofactors of a square matrix A is the adjoint of A .

(b) In Cramer's Rule, the denominator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants.

3 Projects



1 Stochastic Matrices

In Section 2.5, you studied a consumer preference model for competing satellite television companies. The matrix of transition probabilities was

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}.$$

When you were given the initial state matrix X_0 , you observed that the portions of the total population in the three states (subscribing to Company A, subscribing to Company B, and not subscribing) after 1 year was $X_1 = PX_0$.

$$X_0 = \begin{bmatrix} 0.1500 \\ 0.2000 \\ 0.6500 \end{bmatrix}$$

$$X_1 = PX_0 = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} 0.1500 \\ 0.2000 \\ 0.6500 \end{bmatrix} = \begin{bmatrix} 0.2325 \\ 0.2875 \\ 0.4800 \end{bmatrix}$$

After 15 years, the state matrix had nearly reached a steady state.

$$X_{15} = P^{15}X_0 \approx \begin{bmatrix} 0.3333 \\ 0.4756 \\ 0.1911 \end{bmatrix}$$

That is, for large values of n , the product $P^n X$ approached a limit $\bar{X}$, $P\bar{X} = \bar{X}$.

$P\bar{X} = \bar{X} = 1\bar{X}$, so 1 is an *eigenvalue* of P with corresponding *eigenvector* $\bar{X}$. You will study eigenvalues and eigenvectors in more detail in Chapter 7.

-  1. Use a software program or a graphing utility to verify the eigenvalues and eigenvectors of P listed below. That is, show that $P\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, 2$, and 3.

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 0.65, \lambda_3 = 0.55$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} 7 \\ 10 \\ 4 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

2. Let S be the matrix whose columns are the eigenvectors of P . Show that $S^{-1}PS$ is a diagonal matrix D . What are the entries along the diagonal of D ?
3. Show that $P^n = (SDS^{-1})^n = SD^nS^{-1}$. Use this result to calculate X_{15} and verify the result above.

2 The Cayley-Hamilton Theorem

The **characteristic polynomial** of a square matrix A is the determinant $|\lambda I - A|$. If the order of A is n , then the characteristic polynomial $p(\lambda)$ is an n th-degree polynomial in the variable λ .

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

The Cayley-Hamilton Theorem asserts that every square matrix satisfies its characteristic polynomial. That is, for the $n \times n$ matrix A , $p(A) = O$, or

$$A^n + c_{n-1}A^{n-1} + \cdots + c_2A^2 + c_1A + c_0I = O.$$

Note that this is a matrix equation. The $n \times n$ zero matrix is on the right, and the coefficient c_0 is multiplied by the $n \times n$ identity matrix I .

1. Verify the Cayley-Hamilton Theorem for the matrix

$$\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}.$$

2. Verify the Cayley-Hamilton Theorem for the matrix

$$\begin{bmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix}.$$

3. Verify the Cayley-Hamilton Theorem for a general 2×2 matrix A ,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

4. For a nonsingular $n \times n$ matrix A , show that

$$A^{-1} = \frac{1}{c_0}(-A^{n-1} - c_{n-1}A^{n-2} - \cdots - c_2A - c_1I).$$

Use this result to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

5. The Cayley-Hamilton Theorem is useful for calculating powers A^n of the square matrix A . For example, the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\text{is } p(\lambda) = \lambda^2 - 2\lambda - 1.$$

Using the Cayley-Hamilton Theorem,

$$A^2 - 2A - I = O \quad \text{or} \quad A^2 = 2A + I.$$

So, A^2 is written in terms of A and I .

$$A^2 = 2A + I = 2 \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 4 & -1 \end{bmatrix}$$

Similarly, multiplying both sides of the equation $A^2 = 2A + I$ by A gives A^3 in terms of A^2 , A , and I . Moreover, you can write A^3 in terms of A and I by replacing A^2 with $2A + I$, as shown below.

$$A^3 = 2A^2 + A = 2(2A + I) + A = 5A + 2I$$

- (a) Write A^4 in terms of A and I .

- (b) Find A^5 for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

(Hint: Find the characteristic polynomial of A , then use the Cayley-Hamilton Theorem to write A^3 in terms of A^2 , A , and I . Inductively write A^5 in terms of A^2 , A , and I .)

1–3 Cumulative Test

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers in the back of the book.

In Exercises 1 and 2, determine whether the equation is linear in the variables x and y .

1. $\frac{4}{y} - x = 10$

2. $\frac{3}{5}x + \frac{7}{10}y = 2$

In Exercises 3 and 4, use Gaussian elimination to solve the system of linear equations.

3. $\begin{aligned} x - 2y &= 5 \\ 3x + y &= 1 \end{aligned}$

4. $\begin{aligned} 4x_1 + x_2 - 3x_3 &= 11 \\ 2x_1 - 3x_2 + 2x_3 &= 9 \\ x_1 + x_2 + x_3 &= -3 \end{aligned}$



5. Use a software program or a graphing utility to solve the system of linear equations.

$$0.2x - 2.3y + 1.4z - 0.55w = -110.6$$

$$3.4x + 1.3y - 1.7z + 0.45w = 65.4$$

$$0.5x - 4.9y + 1.1z - 1.6w = -166.2$$

$$0.6x + 2.8y - 3.4z + 0.3w = 189.6$$

6. Find the solution set of the system of linear equations represented by the augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & -1 & 4 \end{array} \right]$$

7. Solve the homogeneous linear system corresponding to the coefficient matrix.

$$\left[\begin{array}{cccc} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & -4 \\ -2 & -4 & 1 & -2 \end{array} \right]$$

8. Determine the value(s) of k such that the system is consistent.

$$x + 2y - z = 3$$

$$-x - y + z = 2$$

$$-x + y + z = k$$

9. Solve for x and y in the matrix equation $2A - B = I$, given

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & 2 \\ y & 5 \end{bmatrix}.$$

10. Find $A^T A$ for the matrix $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 4 & 6 \end{bmatrix}$. Show that this product is symmetric.

In Exercises 11–14, find the inverse of the matrix (if it exists).

11. $\begin{bmatrix} -2 & 3 \\ 4 & 6 \end{bmatrix}$ 12. $\begin{bmatrix} -2 & 3 \\ 3 & 6 \end{bmatrix}$ 13. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 14. $\begin{bmatrix} 1 & 1 & 0 \\ -3 & 6 & 5 \\ 0 & 1 & 0 \end{bmatrix}$

In Exercises 15 and 16, use an inverse matrix to solve the system of linear equations.

15. $\begin{aligned} x + 2y &= 0 \\ 3x - 6y &= 8 \end{aligned}$

16. $\begin{aligned} 2x - y &= 6 \\ 2x + y &= 10 \end{aligned}$

17. Find a sequence of elementary matrices whose product is the nonsingular matrix below.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$$

18. Find the determinant of the matrix

$$\begin{bmatrix} 4 & 0 & 3 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 1 & 5 & 1 \\ 1 & 1 & 0 & -3 \end{bmatrix}.$$

19. Find (a) $|A|$, (b) $|B|$, (c) AB , and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

$$A = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$$

20. Find (a) $|A|$ and (b) $|A^{-1}|$.

$$A = \begin{bmatrix} 5 & -2 & -3 \\ -1 & 0 & 4 \\ 6 & -8 & 2 \end{bmatrix}$$

21. If $|A| = 7$ and A is of order 4, then find each determinant.

$$(a) |3A| \quad (b) |A^T| \quad (c) |A^{-1}| \quad (d) |A^3|$$

22. Use the adjoint of

$$A = \begin{bmatrix} 1 & -5 & -1 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

to find A^{-1} .

23. Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{b}$ be the column matrices below.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find constants a , b , and c such that $a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 = \mathbf{b}$.

24. Use a system of linear equations to find the parabola $y = ax^2 + bx + c$ that passes through the points $(-1, 2)$, $(0, 1)$, and $(2, 6)$.

25. Use a determinant to find an equation of the line passing through the points $(1, 4)$ and $(5, -2)$.

26. Use a determinant to find the area of the triangle with vertices $(-2, 2)$, $(8, 2)$, and $(6, -5)$.

27. Determine the currents I_1 , I_2 , and I_3 for the electrical network shown in the figure at the left.

28. A manufacturer produces three different models of a product and ships them to two warehouses. In the matrix

$$A = \begin{bmatrix} 200 & 300 \\ 600 & 350 \\ 250 & 400 \end{bmatrix}$$

a_{ij} represents the number of units of model i that the manufacturer ships to warehouse j . The matrix

$$B = \begin{bmatrix} 12.50 & 9.00 & 21.50 \end{bmatrix}$$

represents the prices of the three models in dollars per unit. Find the product BA and state what each entry of the matrix represents.

29. Let A , B , and C be three nonzero $n \times n$ matrices such that $AC = BC$. Does it follow that $A = B$? If so, provide a proof. If not, provide a counterexample.

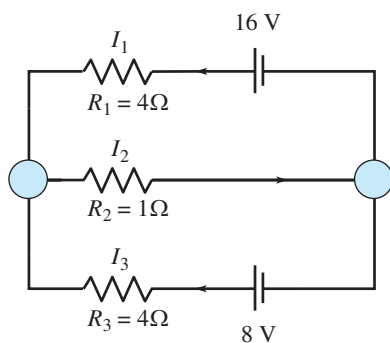


Figure for 27