

Elementary Linear Algebra

8e

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Elementary Linear Algebra
Eighth Edition**Ron Larson**

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A1

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Technology Guide*

*Available online at **CengageBrain.com**.

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- 4.1 Vectors in R^n
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- 4.4 Spanning Sets and Linear Independence
- 4.5 Basis and Dimension
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- 4.8 Applications of Vector Spaces



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4.1 Vectors in R^n

-  Represent a vector as a directed line segment.
-  Perform basic vector operations in R^2 and represent them graphically.
-  Perform basic vector operations in R^n .
-  Write a vector as a linear combination of other vectors.

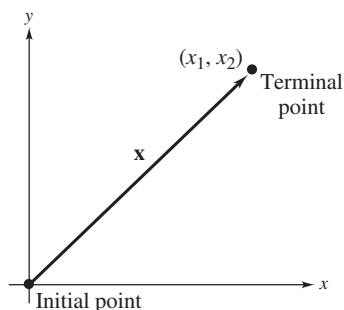
VECTORS IN THE PLANE

In physics and engineering, a *vector* is characterized by two quantities (length and direction) and is represented by a *directed line segment*. In this chapter you will see that the geometric representation can help you understand the more general definition of a vector.

Geometrically, a **vector in the plane** is represented by a **directed line segment** with its **initial point** at the origin and its **terminal point** at (x_1, x_2) , as shown below.

REMARK

The term *vector* derives from the Latin word *vectus*, meaning “to carry.” The idea is that if you were to carry something from the origin to the point (x_1, x_2) , then the trip could be represented by the directed line segment from $(0, 0)$ to (x_1, x_2) . Vectors are represented by lowercase letters set in boldface type (such as **u**, **v**, **w**, and **x**).



The same **ordered pair** used to represent its terminal point also represents the vector. That is, $\mathbf{x} = (x_1, x_2)$. The coordinates x_1 and x_2 are the **components** of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are **equal** if and only if

$$u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

EXAMPLE 1 Vectors in the Plane

- a. To represent $\mathbf{u} = (2, 3)$, draw a directed line segment from the origin to the point $(2, 3)$, as shown in Figure 4.1(a).
- b. To represent $\mathbf{v} = (-1, 2)$, draw a directed line segment from the origin to the point $(-1, 2)$, as shown in Figure 4.1(b).

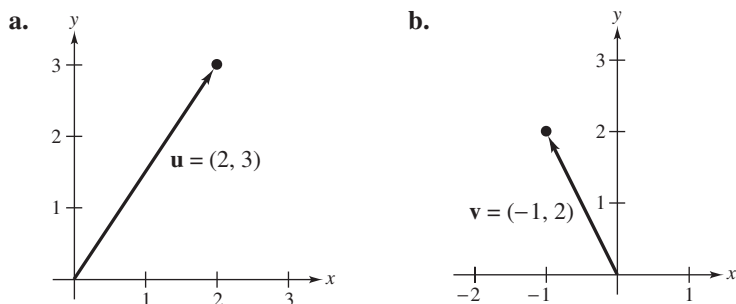


Figure 4.1

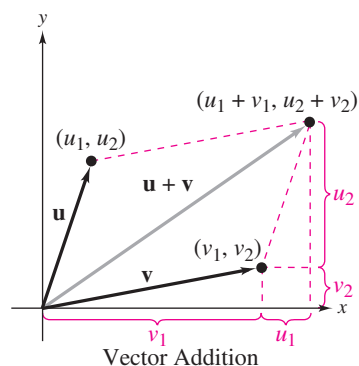


Figure 4.2

VECTOR OPERATIONS

One basic vector operation is **vector addition**. To add two vectors in the plane, add their corresponding components. That is, the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

Geometrically, the sum of two vectors in the plane can be represented by the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides, as shown in Figure 4.2.

In the next example, one of the vectors you will add is the vector $(0, 0)$, the **zero vector**. The zero vector is denoted by $\mathbf{0}$.

EXAMPLE 2

Adding Two Vectors in the Plane

Find each vector sum $\mathbf{u} + \mathbf{v}$.

- $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$
- $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$
- $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

SOLUTION

- $\mathbf{u} + \mathbf{v} = (1, 4) + (2, -2) = (3, 2)$
- $\mathbf{u} + \mathbf{v} = (3, -2) + (-3, 2) = (0, 0) = \mathbf{0}$
- $\mathbf{u} + \mathbf{v} = (2, 1) + (0, 0) = (2, 1)$

Figure 4.3 shows a graphical representation of each sum.

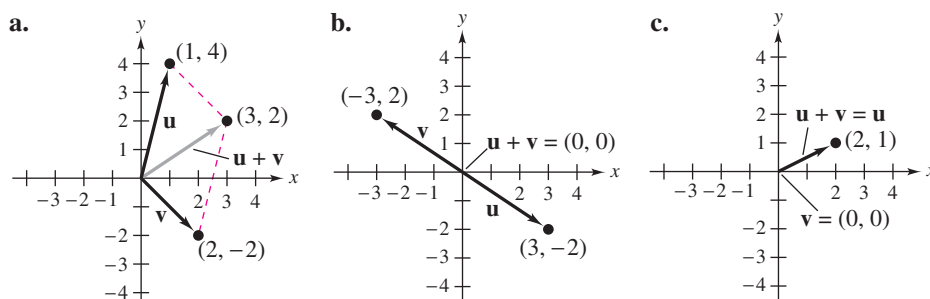


Figure 4.3

Another basic vector operation is **scalar multiplication**. To multiply a vector $\mathbf{v}$ by a scalar c , multiply each of the components of $\mathbf{v}$ by c . That is,

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2).$$

Recall from Chapter 2 that the word *scalar* is used to mean a real number. Historically, this usage arose from the fact that multiplying a vector by a real number changes the “scale” of the vector. For instance, when a vector $\mathbf{v}$ is multiplied by 2, the resulting vector $2\mathbf{v}$ is a vector having the same direction as $\mathbf{v}$ and twice the length. In general, for a scalar c , the vector $c\mathbf{v}$ will be $|c|$ times as long as $\mathbf{v}$. If c is positive, then $c\mathbf{v}$ and $\mathbf{v}$ have the same direction, and if c is negative, then $c\mathbf{v}$ and $\mathbf{v}$ have opposite directions. Figure 4.4 shows this.

The product of a vector $\mathbf{v}$ and the scalar -1 is denoted by

$$-\mathbf{v} = (-1)\mathbf{v}.$$

The vector $-\mathbf{v}$ is the **negative** of $\mathbf{v}$. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

The vector $\mathbf{v}$ is **subtracted** from $\mathbf{u}$ by adding the negative of $\mathbf{v}$.

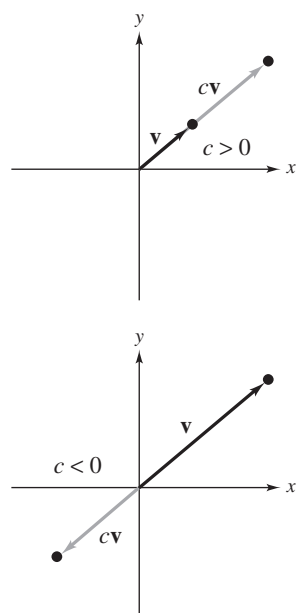


Figure 4.4

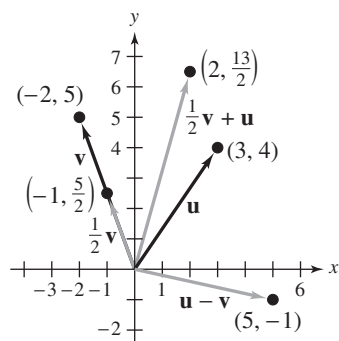


Figure 4.5

EXAMPLE 3**Operations with Vectors in the Plane**

Let $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$. Perform each vector operation.

- a. $\frac{1}{2}\mathbf{v}$ b. $\mathbf{u} - \mathbf{v}$ c. $\frac{1}{2}\mathbf{v} + \mathbf{u}$

SOLUTION

a. $\mathbf{v} = (-2, 5)$, so $\frac{1}{2}\mathbf{v} = (\frac{1}{2}(-2), \frac{1}{2}(5)) = (-1, \frac{5}{2})$.

b. By the definition of vector subtraction, $\mathbf{u} - \mathbf{v} = (3 - (-2), 4 - 5) = (5, -1)$.

c. Using the result of part (a), $\frac{1}{2}\mathbf{v} + \mathbf{u} = (-1, \frac{5}{2}) + (3, 4) = (2, \frac{13}{2})$.

Figure 4.5 shows a graphical representation of these vector operations.

Vector addition and scalar multiplication share many properties with matrix addition and scalar multiplication. The ten properties listed in the next theorem play a fundamental role in linear algebra. In fact, in the next section you will see that it is precisely these ten properties that help define a vector space.

THEOREM 4.1 Properties of Vector Addition and Scalar Multiplication in the Plane

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

- | | |
|--|--|
| 1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane. | Closure under addition |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | Additive identity property |
| 5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | Additive inverse property |
| 6. $c\mathbf{u}$ is a vector in the plane. | Closure under scalar multiplication |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property of multiplication |
| 10. $1(\mathbf{u}) = \mathbf{u}$ | Multiplicative identity property |

REMARK

Note that the associative property of vector addition allows you to write such expressions as $\mathbf{u} + \mathbf{v} + \mathbf{w}$ without ambiguity, because you obtain the same vector sum regardless of which addition is performed first.

PROOF

The proof of each property is straightforward. For example, to prove the associative property of vector addition, write

$$\begin{aligned}
 (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [(u_1, u_2) + (v_1, v_2)] + (w_1, w_2) \\
 &= (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \\
 &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) \\
 &= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \\
 &= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) \\
 &= \mathbf{u} + (\mathbf{v} + \mathbf{w}).
 \end{aligned}$$

Similarly, to prove the right distributive property of scalar multiplication over addition, write

$$\begin{aligned}
 (c + d)\mathbf{u} &= (c + d)(u_1, u_2) \\
 &= ((c + d)u_1, (c + d)u_2) \\
 &= (cu_1 + du_1, cu_2 + du_2) \\
 &= (cu_1, cu_2) + (du_1, du_2) \\
 &= c\mathbf{u} + d\mathbf{u}.
 \end{aligned}$$

The proofs of the other eight properties are left as an exercise. (See Exercise 63.)

VECTORS IN R^n

The discussion of vectors in the plane can be extended to a discussion of vectors in n -space. An **ordered n -tuple** represents a vector in n -space. For instance, an ordered triple has the form (x_1, x_2, x_3) , an ordered quadruple has the form (x_1, x_2, x_3, x_4) , and a general ordered n -tuple has the form $(x_1, x_2, x_3, \dots, x_n)$. The set of all n -tuples is **n -space** and is denoted by R^n .

R^1 = 1-space = set of all real numbers

R^2 = 2-space = set of all ordered pairs of real numbers

R^3 = 3-space = set of all ordered triples of real numbers

$\vdots$

R^n = n -space = set of all ordered n -tuples of real numbers

An n -tuple $(x_1, x_2, x_3, \dots, x_n)$ can be viewed as a **point** in R^n with the x_i 's as its coordinates, or as a **vector** $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ with the x_i 's as its components. As with vectors in the plane (or R^2), two vectors in R^n are **equal** if and only if corresponding components are equal. [In the case of $n = 2$ or $n = 3$, the familiar (x, y) or (x, y, z) notation is used occasionally.]

The sum of two vectors in R^n and the scalar multiple of a vector in R^n are the **standard operations in R^n** and are defined below.

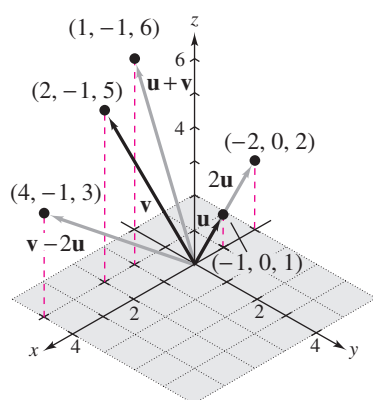


Figure 4.6

Definitions of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u} = (u_1, u_2, u_3, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \dots, v_n)$ be vectors in R^n and let c be a real number. The sum of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots, u_n + v_n)$$

and the **scalar multiple** of $\mathbf{u}$ by c is the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \dots, cu_n).$$

As with 2-space, the **negative** of a vector in R^n is

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \dots, -u_n)$$

and the **difference** of two vectors in R^n is

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \dots, u_n - v_n).$$

The **zero vector** in R^n is denoted by $\mathbf{0} = (0, 0, 0, \dots, 0)$.

TECHNOLOGY

Many graphing utilities and software programs can perform vector addition and scalar multiplication. If you use a graphing utility, then you may verify Example 4(b) as shown below. The **Technology Guide** at CengageBrain.com can help you use technology to perform vector operations.

```
VECTOR:U      3
e1=-1
e2=0
e3=1
2U            [-2 0 2]
```

EXAMPLE 4 Vector Operations in R^3

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Let $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in R^3 . Perform each vector operation.

- a. $\mathbf{u} + \mathbf{v}$ b. $2\mathbf{u}$ c. $\mathbf{v} - 2\mathbf{u}$

SOLUTION

- a. To add two vectors, add their corresponding components.

$$\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$$

- b. To multiply a vector by a scalar, multiply each component by the scalar.

$$2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$$

- c. Using the result of part (b), $\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3)$.

Figure 4.6 shows a graphical representation of these vector operations in R^3 .



William Rowan Hamilton
(1805–1865)

Hamilton is considered to be Ireland's most famous mathematician. In 1828, he published an impressive work on optics entitled *A Theory of Systems of Rays*. In it, Hamilton included some of his own methods for working with systems of linear equations. He also introduced the notion of the characteristic equation of a matrix (see Section 7.1). Hamilton's work led to the development of modern vector notation. We still use his **i**, **j**, and **k** notation for the standard unit vectors in R^3 (see Section 5.1).



The properties of vector addition and scalar multiplication for vectors in R^n listed below are similar to those listed in Theorem 4.1 for vectors in R^2 . Their proofs, based on the definitions of vector addition and scalar multiplication in R^n , are left as an exercise. (See Exercise 64.)

THEOREM 4.2 Properties of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n , and let c and d be scalars.

- | | |
|--|--|
| 1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n . | Closure under addition |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | Additive identity property |
| 5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | Additive inverse property |
| 6. $c\mathbf{u}$ is a vector in R^n . | Closure under scalar multiplication |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property of multiplication |
| 10. $1(\mathbf{u}) = \mathbf{u}$ | Multiplicative identity property |

Using the ten properties from Theorem 4.2, you can perform algebraic manipulations with vectors in R^n in much the same way as you do with real numbers, as demonstrated in the next example.

EXAMPLE 5 Vector Operations in R^4

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in R^4 . Find $\mathbf{x}$ using each equation.

- a. $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$
 b. $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

SOLUTION

- a. Using the properties listed in Theorem 4.2, you have

$$\begin{aligned}
 \mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
 &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
 &= 2(2, -1, 5, 0) - (4, 3, 1, -1) - 3(-6, 2, 0, 3) \\
 &= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
 &= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
 &= (18, -11, 9, -8).
 \end{aligned}$$

- b. Begin by solving for $\mathbf{x}$.

$$\begin{aligned}
 3(\mathbf{x} + \mathbf{w}) &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
 3\mathbf{x} + 3\mathbf{w} &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
 3\mathbf{x} - \mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
 2\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
 \mathbf{x} &= \frac{1}{2}(2\mathbf{u} - \mathbf{v} - 3\mathbf{w})
 \end{aligned}$$

Using the result of part (a),

$$\begin{aligned}
 \mathbf{x} &= \frac{1}{2}(18, -11, 9, -8) \\
 &= \left(9, -\frac{11}{2}, \frac{9}{2}, -4\right).
 \end{aligned}$$



The zero vector $\mathbf{0}$ in R^n is the **additive identity** in R^n . Similarly, the vector $-\mathbf{v}$ is the **additive inverse** of $\mathbf{v}$. The next theorem summarizes several important properties of the additive identity and additive inverse in R^n .

REMARK

Note that in Properties 3 and 5, two different zeros are used, the scalar 0 and the vector $\mathbf{0}$.

THEOREM 4.3 Properties of Additive Identity and Additive Inverse

Let $\mathbf{v}$ be a vector in R^n , and let c be a scalar. Then the properties below are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

PROOF

To prove the first property, assume $\mathbf{v} + \mathbf{u} = \mathbf{v}$. Then Theorem 4.2 justifies the steps below.

$\mathbf{v} + \mathbf{u} = \mathbf{v}$	Given
$(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$	Add $-\mathbf{v}$ to both sides.
$(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{0}$	Additive inverse
$(\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$	Commutative property
$\mathbf{u} + [\mathbf{v} + (-\mathbf{v})] = \mathbf{0}$	Associative property
$\mathbf{u} + \mathbf{0} = \mathbf{0}$	Additive inverse
$\mathbf{u} = \mathbf{0}$	Additive identity

To prove the second property, assume $\mathbf{v} + \mathbf{u} = \mathbf{0}$, and again use Theorem 4.2 to justify the steps below.

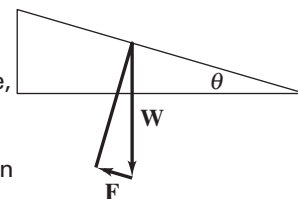
$\mathbf{v} + \mathbf{u} = \mathbf{0}$	Given
$(-\mathbf{v}) + (\mathbf{v} + \mathbf{u}) = (-\mathbf{v}) + \mathbf{0}$	Add $-\mathbf{v}$ to both sides.
$(-\mathbf{v}) + (\mathbf{v} + \mathbf{u}) = -\mathbf{v}$	Additive identity
$[(-\mathbf{v}) + \mathbf{v}] + \mathbf{u} = -\mathbf{v}$	Associative property
$\mathbf{0} + \mathbf{u} = -\mathbf{v}$	Additive inverse
$\mathbf{u} + \mathbf{0} = -\mathbf{v}$	Commutative property
$\mathbf{u} = -\mathbf{v}$	Additive identity

As you gain experience in reading and writing proofs involving vector algebra, you will not need to list as many steps as shown above. For now, however, it is a good idea to list as many steps as possible. The proofs of the other four properties are left as exercises. (See Exercises 65–68.)



LINEAR ALGEBRA APPLIED

Vectors have a wide variety of applications in engineering and the physical sciences. For example, to determine the amount of force required to pull an object up a ramp that has an angle of elevation θ , use the figure at the right.



In the figure, the vector labeled $\mathbf{W}$ represents the weight of the object, and the vector labeled $\mathbf{F}$ represents the required force. Using similar triangles and some trigonometry, the required force is $\mathbf{F} = \mathbf{W} \sin \theta$. (Verify this.)

Anne Kitzman/Shutterstock.com

LINEAR COMBINATIONS OF VECTORS

An important type of problem in linear algebra involves writing one vector $\mathbf{x}$ as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. That is, for scalars $c_1, c_2, \dots, c_n$,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n.$$

The vector $\mathbf{x}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

DISCOVERY

1. Is the vector $(1, 1)$ a linear combination of the vectors $(1, 2)$ and $(-2, -4)$? Graph these vectors and explain your answer geometrically.
2. Similarly, determine whether the vector $(1, 1)$ is a linear combination of the vectors $(1, 2)$ and $(2, 1)$.
3. What is the geometric significance of questions 1 and 2?
4. Is every vector in R^2 a linear combination of the vectors $(1, -2)$ and $(-2, 1)$? Give a geometric explanation for your answer.

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

EXAMPLE 6

Writing a Vector as a Linear Combination of Other Vectors

Let $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in R^3 . Find scalars a, b , and c such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

SOLUTION

Write

$$\begin{aligned} \underbrace{(-1, -2, -2)}_{\mathbf{x}} &= a \underbrace{(0, 1, 4)}_{\mathbf{u}} + b \underbrace{(-1, 1, 2)}_{\mathbf{v}} + c \underbrace{(3, 1, 2)}_{\mathbf{w}} \\ &= (-b + 3c, a + b + c, 4a + 2b + 2c) \end{aligned}$$

and equate corresponding components so that they form the system of three linear equations in a, b , and c shown below.

$$\begin{array}{ll} -b + 3c = -1 & \text{Equation from first component} \\ a + b + c = -2 & \text{Equation from second component} \\ 4a + 2b + 2c = -2 & \text{Equation from third component} \end{array}$$

Solve for a, b , and c to get $a = 1$, $b = -2$, and $c = -1$. As a linear combination of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$,

$$\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}.$$

Use vector addition and scalar multiplication to check this result. 

You will often find it useful to represent a vector $\mathbf{u} = (u_1, u_2, \dots, u_n)$ in R^n as either a $1 \times n$ row matrix (row vector) or an $n \times 1$ column matrix (column vector). This approach is valid because the matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations. That is, the matrix sums

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= [u_1 \ u_2 \ \dots \ u_n] + [v_1 \ v_2 \ \dots \ v_n] \\ &= [u_1 + v_1 \ u_2 + v_2 \ \dots \ u_n + v_n] \end{aligned}$$

and

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

yield the same results as the vector operation of addition,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n). \end{aligned}$$

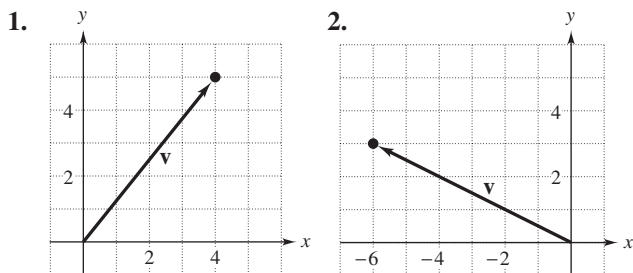
The same argument applies to scalar multiplication. The only difference in each set of notations is how the components (entries) are displayed.

4.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding the Component Form of a Vector In Exercises 1 and 2, find the component form of the vector.



Representing a Vector In Exercises 3–6, use a directed line segment to represent the vector.

3. $\mathbf{u} = (2, -4)$ 4. $\mathbf{v} = (-2, 3)$
 5. $\mathbf{u} = (-3, -4)$ 6. $\mathbf{v} = (-2, -5)$

Finding the Sum of Two Vectors In Exercises 7–10, find the sum of the vectors and illustrate the sum geometrically.

7. $\mathbf{u} = (1, 3)$, $\mathbf{v} = (2, -2)$ 8. $\mathbf{u} = (-1, 4)$, $\mathbf{v} = (4, -3)$
 9. $\mathbf{u} = (2, -3)$, $\mathbf{v} = (-3, -1)$
 10. $\mathbf{u} = (4, -2)$, $\mathbf{v} = (-2, -3)$

Vector Operations In Exercises 11–16, find the vector $\mathbf{v}$ and illustrate the specified vector operations geometrically, where $\mathbf{u} = (-2, 3)$ and $\mathbf{w} = (-3, -2)$.

11. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ 12. $\mathbf{v} = \mathbf{u} + \mathbf{w}$
 13. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ 14. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$
 15. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$ 16. $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

17. For the vector $\mathbf{v} = (2, 1)$, sketch (a) $2\mathbf{v}$, (b) $-3\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.
 18. For the vector $\mathbf{v} = (3, -2)$, sketch (a) $4\mathbf{v}$, (b) $-\frac{1}{2}\mathbf{v}$, and (c) $0\mathbf{v}$.

Vector Operations In Exercises 19–24, let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 2, -1)$, and $\mathbf{w} = (4, 0, -4)$.

19. Find $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. 20. Find $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$.
 21. Find $2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$. 22. Find $5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$.
 23. Find $\mathbf{z}$, where $3\mathbf{u} - 4\mathbf{z} = \mathbf{w}$.
 24. Find $\mathbf{z}$, where $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$.
 25. For the vector $\mathbf{v} = (1, 2, 2)$, sketch (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.
 26. For the vector $\mathbf{v} = (2, 0, 1)$, sketch (a) $-\mathbf{v}$, (b) $2\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.
 27. Determine whether each vector is a scalar multiple of $\mathbf{z} = (3, 2, -5)$.
 (a) $\mathbf{v} = (\frac{9}{2}, 3, -\frac{15}{2})$ (b) $\mathbf{w} = (9, -6, -15)$

28. Determine whether each vector is a scalar multiple of $\mathbf{z} = (\frac{1}{2}, -\frac{2}{3}, \frac{3}{4})$.

- (a) $\mathbf{u} = (6, -4, 9)$
 (b) $\mathbf{v} = (-1, \frac{4}{3}, -\frac{3}{2})$

Vector Operations In Exercises 29–32, find (a) $\mathbf{u} - \mathbf{v}$, (b) $2(\mathbf{u} + 3\mathbf{v})$, and (c) $2\mathbf{v} - \mathbf{u}$.

29. $\mathbf{u} = (4, 0, -3, 5)$, $\mathbf{v} = (0, 2, 5, 4)$
 30. $\mathbf{u} = (0, 4, 3, 4, 4)$, $\mathbf{v} = (6, 8, -3, 3, -5)$
 31. $\mathbf{u} = (-7, 0, 0, 0, 9)$, $\mathbf{v} = (2, -3, -2, 3, 3)$
 32. $\mathbf{u} = (6, -5, 4, 3)$, $\mathbf{v} = (-2, \frac{5}{3}, -\frac{4}{3}, -1)$

 **Vector Operations** In Exercises 33 and 34, use a graphing utility to perform each operation where $\mathbf{u} = (1, 2, -3, 1)$, $\mathbf{v} = (0, 2, -1, -2)$, and $\mathbf{w} = (2, -2, 1, 3)$.

33. (a) $\mathbf{u} + 2\mathbf{v}$ 34. (a) $\mathbf{v} + 3\mathbf{w}$
 (b) $\mathbf{w} - 3\mathbf{u}$ (b) $2\mathbf{w} - \frac{1}{2}\mathbf{u}$
 (c) $4\mathbf{v} + \frac{1}{2}\mathbf{u} - \mathbf{w}$ (c) $\frac{1}{2}(4\mathbf{v} - 3\mathbf{u} + \mathbf{w})$

Solving a Vector Equation In Exercises 35–38, solve for $\mathbf{w}$, where $\mathbf{u} = (1, -1, 0, 1)$ and $\mathbf{v} = (0, 2, 3, -1)$.

35. $3\mathbf{w} = \mathbf{u} - 2\mathbf{v}$ 36. $\mathbf{w} + \mathbf{u} = -\mathbf{v}$
 37. $\frac{1}{2}\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$ 38. $\mathbf{w} + 3\mathbf{v} = -2\mathbf{u}$

Solving a Vector Equation In Exercises 39 and 40, find $\mathbf{w}$ such that $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \mathbf{0}$.

39. $\mathbf{u} = (0, 2, 7, 5)$, $\mathbf{v} = (-3, 1, 4, -8)$
 40. $\mathbf{u} = (-6, 0, 2, 0)$, $\mathbf{v} = (5, -3, 0, 1)$

Writing a Linear Combination In Exercises 41–46, write $\mathbf{v}$ as a linear combination of $\mathbf{u}$ and $\mathbf{w}$, if possible, where $\mathbf{u} = (1, 2)$ and $\mathbf{w} = (1, -1)$.

41. $\mathbf{v} = (2, 1)$ 42. $\mathbf{v} = (0, 3)$
 43. $\mathbf{v} = (3, 3)$ 44. $\mathbf{v} = (1, -1)$
 45. $\mathbf{v} = (-1, -2)$ 46. $\mathbf{v} = (1, -4)$

Writing a Linear Combination In Exercises 47–50, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.

47. $\mathbf{v} = (10, 1, 4)$, $\mathbf{u}_1 = (2, 3, 5)$, $\mathbf{u}_2 = (1, 2, 4)$, $\mathbf{u}_3 = (-2, 2, 3)$
 48. $\mathbf{v} = (-1, 7, 2)$, $\mathbf{u}_1 = (1, 3, 5)$, $\mathbf{u}_2 = (2, -1, 3)$, $\mathbf{u}_3 = (-3, 2, -4)$
 49. $\mathbf{v} = (0, 5, 3, 0)$, $\mathbf{u}_1 = (1, 1, 2, 2)$, $\mathbf{u}_2 = (2, 3, 5, 6)$, $\mathbf{u}_3 = (-3, 1, -4, 2)$
 50. $\mathbf{v} = (7, 2, 5, -3)$, $\mathbf{u}_1 = (2, 1, 1, 2)$, $\mathbf{u}_2 = (-3, 3, 4, -5)$, $\mathbf{u}_3 = (-6, 3, 1, 2)$

Writing a Linear Combination In Exercises 51 and 52, write the third column of the matrix as a linear combination of the first two columns, if possible.

51. $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$

52. $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 7 \end{bmatrix}$



Writing a Linear Combination In Exercises 53 and 54, use a software program or a graphing utility to write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, and $\mathbf{u}_5$. Then verify your solution.

53. $\mathbf{v} = (5, 3, -11, 11, 9)$ 54. $\mathbf{v} = (5, 8, 7, -2, 4)$

$\mathbf{u}_1 = (1, 2, -3, 4, -1)$ $\mathbf{u}_1 = (1, 1, -1, 2, 1)$

$\mathbf{u}_2 = (1, 2, 0, 2, 1)$ $\mathbf{u}_2 = (2, 1, 2, -1, 1)$

$\mathbf{u}_3 = (0, 1, 1, 1, -4)$ $\mathbf{u}_3 = (1, 2, 0, 1, 2)$

$\mathbf{u}_4 = (2, 1, -1, 2, 1)$ $\mathbf{u}_4 = (0, 2, 0, 1, -4)$

$\mathbf{u}_5 = (0, 2, 2, -1, -1)$ $\mathbf{u}_5 = (1, 1, 2, -1, 2)$

Writing a Linear Combination In Exercises 55 and 56, the zero vector $\mathbf{0} = (0, 0, 0)$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ because $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$. This is the *trivial* solution. Find a *nontrivial* way of writing $\mathbf{0}$ as a linear combination of the three vectors, if possible.

55. $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (-1, 1, 2)$, $\mathbf{v}_3 = (0, 1, 4)$

56. $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (-1, 1, 2)$, $\mathbf{v}_3 = (0, 1, 3)$

True or False? In Exercises 57 and 58, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

57. (a) Two vectors in R^n are equal if and only if their corresponding components are equal.

(b) The vector $-\mathbf{v}$ is the additive identity of $\mathbf{v}$.

58. (a) To subtract two vectors in R^n , subtract their corresponding components.

(b) The zero vector $\mathbf{0}$ in R^n is the additive inverse of a vector.

59. **Writing** Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in n variables. Designate the columns of A as $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. When $\mathbf{b}$ is a linear combination of these n column vectors, explain why this implies that the linear system is consistent. What can you conclude about the linear system when $\mathbf{b}$ is not a linear combination of the columns of A ?

60. **Writing** How could you describe vector subtraction geometrically? What is the relationship between vector subtraction and the basic vector operations of addition and scalar multiplication?

61. Illustrate properties 1–10 of Theorem 4.2 for $\mathbf{u} = (2, -1, 3, 6)$, $\mathbf{v} = (1, 4, 0, 1)$, $\mathbf{w} = (3, 0, 2, 0)$, $c = 5$, and $d = -2$.

62. **CAPSTONE** Consider the vectors $\mathbf{u} = (3, -4)$ and $\mathbf{v} = (9, 1)$.

(a) Use directed line segments to represent each vector graphically.

(b) Find $\mathbf{u} + \mathbf{v}$.

(c) Find $2\mathbf{v} - \mathbf{u}$.

(d) Write $\mathbf{w} = (39, 0)$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

63. **Proof** Complete the proof of Theorem 4.1.

64. **Proof** Prove each property of vector addition and scalar multiplication from Theorem 4.2.

Proof In Exercises 65–68, complete the proofs of the remaining properties of Theorem 4.3 by supplying the justification for each step. Use the properties of vector addition and scalar multiplication from Theorem 4.2.

65. Property 3: $0\mathbf{v} = \mathbf{0}$

$0\mathbf{v} = (0 + 0)\mathbf{v}$ a. _____

$0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ b. _____

$0\mathbf{v} + (-0\mathbf{v}) = (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v})$ c. _____

$\mathbf{0} = 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v}))$ d. _____

$\mathbf{0} = 0\mathbf{v} + \mathbf{0}$ e. _____

$\mathbf{0} = 0\mathbf{v}$ f. _____

66. Property 4: $c\mathbf{0} = \mathbf{0}$

$c\mathbf{0} = c(\mathbf{0} + \mathbf{0})$ a. _____

$c\mathbf{0} = c\mathbf{0} + c\mathbf{0}$ b. _____

$c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$ c. _____

$\mathbf{0} = c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0}))$ d. _____

$\mathbf{0} = c\mathbf{0} + \mathbf{0}$ e. _____

$\mathbf{0} = c\mathbf{0}$ f. _____

67. Property 5: If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$. If $c = 0$, then you are done. If $c \neq 0$, then c^{-1} exists, and you have

$c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0}$ a. _____

$(c^{-1}c)\mathbf{v} = \mathbf{0}$ b. _____

$1\mathbf{v} = \mathbf{0}$ c. _____

$\mathbf{v} = \mathbf{0}$ d. _____

68. Property 6: $-(-\mathbf{v}) = \mathbf{v}$

$-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ a. _____

$-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$ b. _____

$-(-\mathbf{v}) + (-\mathbf{v}) + \mathbf{v} = \mathbf{v} + (-\mathbf{v}) + \mathbf{v}$ c. _____

$-(-\mathbf{v}) + ((-\mathbf{v}) + \mathbf{v}) = \mathbf{v} + ((-\mathbf{v}) + \mathbf{v})$ d. _____

$-(-\mathbf{v}) + \mathbf{0} = \mathbf{v} + \mathbf{0}$ e. _____

$-(-\mathbf{v}) = \mathbf{v}$ f. _____

4.2 Vector Spaces

-  Define a vector space and recognize some important vector spaces.
-  Show that a given set is not a vector space.

DEFINITION OF A VECTOR SPACE

Theorem 4.2 lists ten properties of vector addition and scalar multiplication in R^n . Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, and functions) also share these ten properties. Any set that satisfies these properties (or **axioms**) is called a **vector space**, and the objects in the set are **vectors**.

It is important to realize that the definition of a vector space below is precisely that—a *definition*. You do not need to prove anything because you are simply listing the axioms required of vector spaces. This type of definition is an **abstraction** because you are abstracting a collection of properties from a particular setting, R^n , to form the axioms for a more general setting.

Definition of a Vector Space

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d , then V is a **vector space**.

Addition:

- | | |
|--|------------------------|
| 1. $\mathbf{u} + \mathbf{v}$ is in V . | Closure under addition |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property |
| 3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ | Associative property |
| 4. V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V , $\mathbf{u} + \mathbf{0} = \mathbf{u}$. | Additive identity |
| 5. For every $\mathbf{u}$ in V , there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. | Additive inverse |

Scalar Multiplication:

- | | |
|---|-------------------------------------|
| 6. $c\mathbf{u}$ is in V . | Closure under scalar multiplication |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property |
| 10. $1(\mathbf{u}) = \mathbf{u}$ | Scalar identity |

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space V , be sure that all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

The first two examples of vector spaces should not be surprising. They are, in fact, the models used to form the ten vector space axioms.

EXAMPLE 1

R^2 with the Standard Operations Is a Vector Space

The set of all ordered pairs of real numbers R^2 with the standard operations is a vector space. To verify this, look back at Theorem 4.1. Vectors in this space have the form

$$\mathbf{v} = (v_1, v_2).$$

REMARK

From Example 2 you can conclude that R^1 , the set of real numbers (with the usual operations of addition and multiplication), is a vector space.

REMARK

In the same way you are able to show that the set of all 2×3 matrices is a vector space, you can show that the set of all $m \times n$ matrices, denoted by $M_{m,n}$, is a vector space.

EXAMPLE 2 **R^n with the Standard Operations Is a Vector Space**

The set of all ordered n -tuples of real numbers R^n with the standard operations is a vector space. Theorem 4.2 verifies this. Vectors in this space are of the form

$$\mathbf{v} = (v_1, v_2, v_3, \dots, v_n).$$

The next three examples describe vector spaces in which the basic set V does not consist of ordered n -tuples. Each example describes the set V and defines the two vector operations. To show that the set is a vector space, you must verify all ten axioms.

EXAMPLE 3**The Vector Space of All 2×3 Matrices**

Show that the set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

SOLUTION

If A and B are 2×3 matrices and c is a scalar, then $A + B$ and cA are also 2×3 matrices. The set is, therefore, closed under matrix addition and scalar multiplication. Moreover, the other eight vector space axioms follow directly from Theorems 2.1 and 2.2 (see Section 2.2). So, the set is a vector space. Vectors in this space have the form

$$\mathbf{a} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

EXAMPLE 4**The Vector Space of All Polynomials of Degree 2 or Less**

Let P_2 be the set of all polynomials of the form $p(x) = a_0 + a_1x + a_2x^2$, where a_0 , a_1 , and a_2 are real numbers. The *sum* of two polynomials $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ is defined in the usual way,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and the *scalar multiple* of $p(x)$ by the scalar c is defined by

$$cp(x) = ca_0 + ca_1x + ca_2x^2.$$

Show that P_2 is a vector space.

SOLUTION

Verification of each of the ten vector space axioms is a straightforward application of the properties of real numbers. For example, the set of real numbers is closed under addition, so it follows that $a_0 + b_0$, $a_1 + b_1$, and $a_2 + b_2$ are real numbers, and

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

is in the set P_2 because it is a polynomial of degree 2 or less. So, P_2 is closed under addition. To verify the commutative property of addition, write

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) \\ &= q(x) + p(x). \end{aligned}$$

Can you see where the commutative property of addition of real numbers was used? The zero vector in this space is the zero polynomial $\mathbf{0}(x) = 0 + 0x + 0x^2$. Verify the other vector space axioms to show that P_2 is a vector space.

REMARK

Even though the zero polynomial $\mathbf{0}(x) = 0$ has no degree, P_2 is often called the set of all polynomials of degree 2 or less.

P_n is defined as the set of all polynomials of degree n or less (together with the zero polynomial). The procedure used to verify that P_2 is a vector space can be extended to show that P_n , with the usual operations of polynomial addition and scalar multiplication, is a vector space.

EXAMPLE 5

The Vector Space of Continuous Functions (Calculus)

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Let $C(-\infty, \infty)$ be the set of all real-valued continuous functions defined on the entire real line. This set consists of all polynomial functions and all other continuous functions on the entire real line. For example, $f(x) = \sin x$ and $g(x) = e^x$ are members of this set.

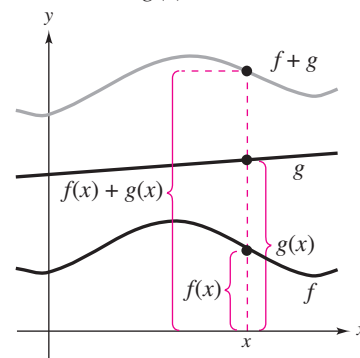
Addition is defined by

$$(f + g)(x) = f(x) + g(x)$$

as shown at the right. Scalar multiplication is defined by

$$(cf)(x) = c[f(x)].$$

Show that $C(-\infty, \infty)$ is a vector space.



SOLUTION

To verify that the set $C(-\infty, \infty)$ is closed under addition and scalar multiplication, use a result from calculus—the sum of two continuous functions is continuous and the product of a scalar and a continuous function is continuous. To verify that the set $C(-\infty, \infty)$ has an additive identity, consider the function f_0 that has a value of zero for all x , that is,

$$f_0(x) = 0, \quad x \text{ is any real number.}$$

This function is continuous on the entire real line (its graph is simply the line $y = 0$), which means that it is in the set $C(-\infty, \infty)$. Moreover, if f is any other function that is continuous on the entire real line, then

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x).$$

This shows that f_0 is the additive identity in $C(-\infty, \infty)$. The verification of the other vector space axioms is left to you. 

The summary below lists some important vector spaces frequently referenced in the remainder of this text. The operations are the standard operations in each case.

Summary of Important Vector Spaces

- R = set of all real numbers
- R^2 = set of all ordered pairs
- R^3 = set of all ordered triples
- R^n = set of all n -tuples
- $C(-\infty, \infty)$ = set of all continuous functions defined on the real number line
- $C[a, b]$ = set of all continuous functions defined on a closed interval $[a, b]$, where $a \neq b$
- P = set of all polynomials
- P_n = set of all polynomials of degree $\leq n$ (together with the zero polynomial)
- $M_{m,n}$ = set of all $m \times n$ matrices
- $M_{n,n}$ = set of all $n \times n$ square matrices

You have seen the versatility of the concept of a vector space. For instance, a vector can be a real number, an n -tuple, a matrix, a polynomial, a continuous function, and so on. But what is the purpose of this abstraction, and why bother to define it? There are several reasons, but the most important reason applies to efficiency. Once a theorem has been proved for an abstract vector space, you need not give separate proofs for n -tuples, matrices, polynomials, or other forms. Simply point out that the theorem is true for any vector space, regardless of the form the vectors have. Theorem 4.4 illustrates this process.

THEOREM 4.4 Properties of Scalar Multiplication

Let $\mathbf{v}$ be any element of a vector space V , and let c be any scalar. Then the properties below are true.

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$

PROOF

To prove these properties, use the appropriate vector space axioms. For example, to prove the second property, note from axiom 4 that $\mathbf{0} = \mathbf{0} + \mathbf{0}$. This allows you to write the steps below.

$$\begin{aligned}
 c\mathbf{0} &= c(\mathbf{0} + \mathbf{0}) && \text{Additive identity} \\
 c\mathbf{0} &= c\mathbf{0} + c\mathbf{0} && \text{Left distributive property} \\
 c\mathbf{0} + (-c\mathbf{0}) &= (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) && \text{Add } -c\mathbf{0} \text{ to both sides.} \\
 c\mathbf{0} + (-c\mathbf{0}) &= c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})] && \text{Associative property} \\
 \mathbf{0} &= c\mathbf{0} + \mathbf{0} && \text{Additive inverse} \\
 \mathbf{0} &= c\mathbf{0} && \text{Additive identity}
 \end{aligned}$$

To prove the third property, let $c\mathbf{v} = \mathbf{0}$. To show that this implies either $c = 0$ or $\mathbf{v} = \mathbf{0}$, assume that $c \neq 0$. (When $c = 0$, you have nothing more to prove.) Now, $c \neq 0$, so you can use the reciprocal $1/c$ to show that $\mathbf{v} = \mathbf{0}$, as shown below.

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{c}\right)(c\mathbf{v}) = \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0}$$

Note that the last step uses Property 2 (the one you just proved). The proofs of the first and fourth properties are left as exercises. (See Exercises 51 and 52.)



LINEAR ALGEBRA APPLIED

In a mass-spring system, motion is assumed to occur in only the vertical direction. That is, the system has one *degree of freedom*. When the mass is pulled downward and then released, the system will oscillate. If there are no forces present to slow or stop the oscillation, then the system is undamped and will oscillate indefinitely. Applying Newton's Second Law of Motion to the mass yields the second order differential equation

$$x'' + \omega^2 x = 0$$

where x is the displacement at time t , and ω is a fixed constant called the *natural frequency* of the system. The general solution of this differential equation is

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where a_1 and a_2 are arbitrary constants. (Verify this.) In Exercise 45, you are asked to show that the set of all functions $x(t)$ is a vector space.

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SETS THAT ARE NOT VECTOR SPACES

The remaining examples in this section describe some sets (with operations) that *do not* form vector spaces. To show that a set is not a vector space, you need only find one axiom that is not satisfied.

REMARK

Notice that a single failure of one of the ten vector space axioms suffices to show that a set is not a vector space.

REMARK

The set of all second-degree polynomials is also not closed under scalar multiplication. (Verify this.)

EXAMPLE 6

The Set of Integers Is Not a Vector Space

The set of all integers (with the standard operations) does not form a vector space because it is not closed under scalar multiplication. For example,

$$\frac{1}{2}(1) = \frac{1}{2}.$$

Scalar
Integer
Noninteger

In Example 4, it was shown that the set of all polynomials of degree 2 or less forms a vector space. You will now see that the set of all polynomials whose degree is exactly 2 does not form a vector space.

EXAMPLE 7

The Set of Second-Degree Polynomials Is Not a Vector Space

The set of all second-degree polynomials is not a vector space because it is not closed under addition. To see this, consider the second-degree polynomials $p(x) = x^2$ and $q(x) = 1 + x - x^2$, whose sum is the first-degree polynomial $p(x) + q(x) = 1 + x$.

The sets in Examples 6 and 7 are not vector spaces because they fail one or both closure axioms. In the next example, you will look at a set that passes both tests for closure but still fails to be a vector space.

EXAMPLE 8

A Set That Is Not a Vector Space

Let $V = \mathbb{R}^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication listed below.

$$c(x_1, x_2) = (cx_1, 0)$$

Show that V is not a vector space.

SOLUTION

In this example, the operation of scalar multiplication is nonstandard. For instance, the product of the scalar 2 and the ordered pair $(3, 4)$ does not equal $(6, 8)$. Instead, the second component of the product is 0,

$$2(3, 4) = (2 \cdot 3, 0) = (6, 0).$$

This example is interesting because it satisfies the first nine axioms of the definition of a vector space (show this). In attempting to verify the tenth axiom, the nonstandard definition of scalar multiplication gives you

$$1(1, 1) = (1, 0) \neq (1, 1).$$

The tenth axiom is not satisfied and the set (together with the two operations) is not a vector space.

Do not be confused by the notation used for scalar multiplication in Example 8. In writing $c(x_1, x_2) = (cx_1, 0)$, the scalar multiple of (x_1, x_2) by c is *defined* to be $(cx_1, 0)$ in this example.

4.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Describing the Additive Identity In Exercises 1–6, describe the zero vector (the additive identity) of the vector space.

1. R^4
2. $C[-1, 0]$
3. $M_{4,3}$
4. $M_{5,1}$
5. P_3
6. $M_{2,2}$

Describing the Additive Inverse In Exercises 7–12, describe the additive inverse of a vector in the vector space.

7. R^3
8. $C(-\infty, \infty)$
9. $M_{2,3}$
10. $M_{1,4}$
11. P_4
12. $M_{5,5}$

Testing for a Vector Space In Exercises 13–36, determine whether the set, together with the standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

13. $M_{4,6}$
14. $M_{1,1}$
15. The set of all third-degree polynomials
16. The set of all fifth-degree polynomials
17. The set of all first-degree polynomial functions ax , $a \neq 0$, whose graphs pass through the origin
18. The set of all first-degree polynomial functions $ax + b$, $a, b \neq 0$, whose graphs *do not* pass through the origin
19. The set of all polynomials of degree four or less
20. The set of all quadratic functions whose graphs pass through the origin
21. The set
 $\{(x, y): x \geq 0, y \text{ is a real number}\}$
22. The set
 $\{(x, y): x \geq 0, y \geq 0\}$
23. The set
 $\{(x, x): x \text{ is a real number}\}$
24. The set
 $\{(x, \frac{1}{2}x): x \text{ is a real number}\}$
25. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$
26. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$$

27. The set of all 3×3 matrices of the form

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$$

28. The set of all 3×3 matrices of the form

$$\begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

29. The set of all 4×4 matrices of the form

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 1 \end{bmatrix}$$

30. The set of all 4×4 matrices of the form

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix}$$

31. The set of all 2×2 singular matrices
32. The set of all 2×2 nonsingular matrices
33. The set of all 2×2 diagonal matrices
34. The set of all 3×3 upper triangular matrices
35. $C[0, 1]$, the set of all continuous functions defined on the interval $[0, 1]$
36. $C[-1, 1]$, the set of all continuous functions defined on the interval $[-1, 1]$
37. Let V be the set of all positive real numbers. Determine whether V is a vector space with the operations shown below.
 $x + y = xy$ Addition
 $cx = x^c$ Scalar multiplication
 If it is, verify each vector space axiom; if it is not, state all vector space axioms that fail.
38. Determine whether the set R^2 with the operations
 $(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2)$
 and
 $c(x_1, y_1) = (cx_1, cy_1)$
 is a vector space. If it is, verify each vector space axiom; if it is not, state all vector space axioms that fail.
39. **Proof** Prove in full detail that the set $\{(x, 2x): x \text{ is a real number}\}$, with the standard operations in R^2 , is a vector space.

40. **Proof** Prove in full detail that $M_{2,2}$, with the standard operations, is a vector space.

41. Rather than use the standard definitions of addition and scalar multiplication in R^2 , let these two operations be defined as shown below.

$$(a) \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c(x, y) = (cx, y)$$

$$(b) \quad (x_1, y_1) + (x_2, y_2) = (x_1, 0) \\ c(x, y) = (cx, cy)$$

$$(c) \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c(x, y) = (\sqrt{cx}, \sqrt{cy})$$

With each of these new definitions, is R^2 a vector space? Justify your answers.

42. Rather than use the standard definitions of addition and scalar multiplication in R^3 , let these two operations be defined as shown below.

$$(a) \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ c(x, y, z) = (cx, cy, 0)$$

$$(b) \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) = (0, 0, 0) \\ c(x, y, z) = (cx, cy, cz)$$

$$(c) \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\ c(x, y, z) = (cx, cy, cz)$$

$$(d) \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\ c(x, y, z) = (cx + c - 1, cy + c - 1, cz + c - 1)$$

With each of these new definitions, is R^3 a vector space? Justify your answers.

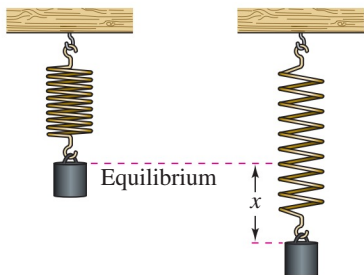
43. Prove that in a given vector space V , the zero vector is unique.

44. Prove that in a given vector space V , the additive inverse of a vector is unique.

45. **Mass-Spring System** The mass in a mass-spring system (see figure) is pulled downward and then released, causing the system to oscillate according to

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where x is the displacement at time t , a_1 and a_2 are arbitrary constants, and ω is a fixed constant. Show that the set of all functions $x(t)$ is a vector space.



46. CAPSTONE

- (a) Describe the conditions under which a set may be classified as a vector space.
- (b) Give an example of a set that is a vector space and an example of a set that is not a vector space.

47. **Proof** Complete the proof of the cancellation property of vector addition by justifying each step.

Prove that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.

$$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$$

$$(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w}) \quad a. \underline{\hspace{2cm}}$$

$$\mathbf{u} + (\mathbf{w} + (-\mathbf{w})) = \mathbf{v} + (\mathbf{w} + (-\mathbf{w})) \quad b. \underline{\hspace{2cm}}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0} \quad c. \underline{\hspace{2cm}}$$

$$\mathbf{u} = \mathbf{v} \quad d. \underline{\hspace{2cm}}$$

48. Let R^∞ be the set of all infinite sequences of real numbers, with the operations

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3, \dots) + (v_1, v_2, v_3, \dots) \\ = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots)$$

and

$$c\mathbf{u} = c(u_1, u_2, u_3, \dots) \\ = (cu_1, cu_2, cu_3, \dots)$$

Determine whether R^∞ is a vector space. If it is, verify each vector space axiom; if it is not, state all vector space axioms that fail.

True or False? In Exercises 49 and 50, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

49. (a) A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.
- (b) The set of all integers with the standard operations is a vector space.
- (c) The set of all ordered triples (x, y, z) of real numbers, where $y \geq 0$, with the standard operations on R^3 is a vector space.
50. (a) To show that a set is not a vector space, it is sufficient to show that just one axiom is not satisfied.
- (b) The set of all first-degree polynomials with the standard operations is a vector space.
- (c) The set of all pairs of real numbers of the form $(0, y)$, with the standard operations on R^2 , is a vector space.

51. **Proof** Prove Property 1 of Theorem 4.4.

52. **Proof** Prove Property 4 of Theorem 4.4.

4.3 Subspaces of Vector Spaces

-  Determine whether a subset W of a vector space V is a subspace of V .
-  Determine subspaces of R^n .

SUBSPACES

In many applications in linear algebra, vector spaces occur as **subspaces** of larger spaces. For instance, you will see that the solution set of a homogeneous system of linear equations in n variables is a subspace of R^n . (See Theorem 4.16.)

A nonempty subset of a vector space is a subspace when it is a vector space with the *same* operations defined in the original vector space, as stated in the next definition.

REMARK

Note that if W is a subspace of V , then it must be closed under the operations inherited from V .

Definition of a Subspace of a Vector Space

A nonempty subset W of a vector space V is a **subspace** of V when W is a vector space under the operations of addition and scalar multiplication defined in V .

EXAMPLE 1 A Subspace of R^3

Show that the set $W = \{(x_1, 0, x_3) : x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of R^3 with the standard operations.

SOLUTION

The set W is nonempty because it contains the zero vector $(0, 0, 0)$.

Graphically, the set W can be interpreted as the xz -plane, as shown in Figure 4.7. The set W is closed under addition because the sum of any two vectors in the xz -plane must also lie in the xz -plane. That is, if $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ are in W , then their sum $(x_1 + y_1, 0, x_3 + y_3)$ is also in W . Similarly, to see that W is closed under scalar multiplication, let $(x_1, 0, x_3)$ be in W and let c be a scalar. Then $c(x_1, 0, x_3) = (cx_1, 0, cx_3)$ has zero as its second component and must be in W . The verifications of the other eight vector space axioms are left to you.

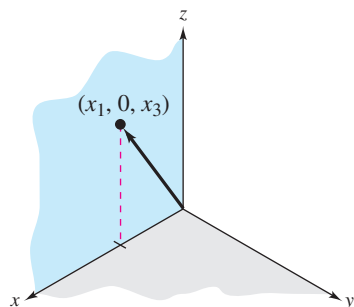


Figure 4.7

To establish that a set W is a vector space, you must verify all ten vector space axioms. If W is a nonempty subset of a larger vector space V (and the operations defined on W are the *same* as those defined on V), however, then most of the ten properties are *inherited* from the larger space and need no verification. The next theorem states that it is sufficient to test for closure in order to establish that a nonempty subset of a vector space is a subspace.

THEOREM 4.5 Test for a Subspace

If W is a nonempty subset of a vector space V , then W is a subspace of V if and only if the two closure conditions listed below hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W , then $\mathbf{u} + \mathbf{v}$ is in W .
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W .

PROOF

The proof of this theorem in one direction is straightforward. That is, if W is a subspace of V , then W is a vector space and must be closed under addition and scalar multiplication.

REMARK

Note that if W is a subspace of a vector space V , then both W and V must have the same zero vector $\mathbf{0}$. (In Exercise 55, you are asked to prove this.)

To prove the theorem in the other direction, assume that W is closed under addition and scalar multiplication. Note that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in W , then they are also in V . Consequently, vector space axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically. W is closed under addition and scalar multiplication, so it follows that for any $\mathbf{v}$ in W and scalar $c = 0$, $c\mathbf{v} = \mathbf{0}$, and $(-1)\mathbf{v} = -\mathbf{v}$ both lie in W , which satisfies axioms 4 and 5. 

A subspace of a vector space is also a vector space, so it must contain the zero vector. In fact, the simplest subspace of a vector space V is the one consisting of only the zero vector, $W = \{\mathbf{0}\}$. This subspace is the **zero subspace**. Another subspace of V is V itself. Every vector space contains these two trivial subspaces, and subspaces other than these are called **proper** (or nontrivial) subspaces.

EXAMPLE 2**A Subspace of $M_{2,2}$**

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

SOLUTION

Recall that a square matrix is *symmetric* when it is equal to its own transpose. The set $M_{2,2}$ is a vector space, so you only need to show that W (a subset of $M_{2,2}$) satisfies the conditions of Theorem 4.5. Begin by observing that W is *nonempty*. W is closed under addition because for matrices A_1 and A_2 in W , $A_1 = A_1^T$ and $A_2 = A_2^T$, which implies that

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2.$$

So, if A_1 and A_2 are symmetric matrices of order 2, then so is $A_1 + A_2$. Similarly, W is closed under scalar multiplication because $A = A^T$ implies that $(cA)^T = cA^T = cA$. If A is a symmetric matrix of order 2, then so is cA . 

The result of Example 2 can be generalized. That is, for any positive integer n , the set of symmetric matrices of order n is a subspace of the vector space $M_{n,n}$ with the standard operations. The next example describes a subset of $M_{n,n}$ that is not a subspace.

EXAMPLE 3**The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$**

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.

SOLUTION

By Theorem 4.5, to show that a subset W is not a subspace, show that W is empty, W is not closed under addition, or W is not closed under scalar multiplication. In this example, W is nonempty and closed under scalar multiplication, but it is not closed under addition. To see this, let A and B be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A and B are both singular (noninvertible), but their sum

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is nonsingular (invertible). So W is not closed under addition, and by Theorem 4.5, W is not a subspace of $M_{2,2}$. 

EXAMPLE 4**A Subset of R^2 That Is Not a Subspace**

Show that $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2 .

SOLUTION

This set is nonempty and closed under addition. It is not, however, closed under scalar multiplication. To see this, note that $(1, 1)$ is in W , but the scalar multiple $(-1)(1, 1) = (-1, -1)$ is not in W . So W is not a subspace of R^2 .

You will often encounter sequences of nested subspaces. For example, consider the vector spaces $P_0, P_1, P_2, P_3, \dots, P_n$, where P_k is the set of all polynomials of degree less than or equal to k , with the standard operations. You can write $P_0 \subset P_1 \subset P_2 \subset P_3 \subset \dots \subset P_n$. If $j \leq k$, then P_j is a subspace of P_k . (In Exercise 45, you are asked to show this.) Example 5 describes another nesting of subspaces.

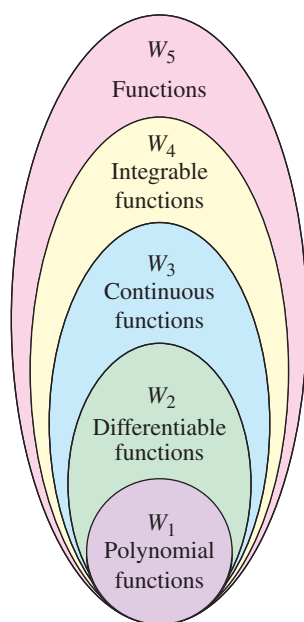


Figure 4.8

EXAMPLE 5**Subspaces of Functions (Calculus)**

Let W_5 be the vector space of all functions defined on $[0, 1]$, and let W_1, W_2, W_3 , and W_4 be defined as shown below.

W_1 = set of all polynomial functions that are defined on $[0, 1]$

W_2 = set of all functions that are differentiable on $[0, 1]$

W_3 = set of all functions that are continuous on $[0, 1]$

W_4 = set of all functions that are integrable on $[0, 1]$

Show that $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$ and that W_i is a subspace of W_j for $i \leq j$.

SOLUTION

From calculus you know that every polynomial function is differentiable on $[0, 1]$. So, $W_1 \subset W_2$. Moreover, every differentiable function is continuous, every continuous function is integrable, and every integrable function is a function, which means that $W_2 \subset W_3 \subset W_4 \subset W_5$. So, you have $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$, as shown in Figure 4.8. It is left to you to show that W_i is a subspace of W_j for $i \leq j$. (See Exercise 46.)

As implied in Example 5, if U, V , and W are vector spaces such that W is a subspace of V and V is a subspace of U , then W is also a subspace of U . The next theorem states that the intersection of two subspaces is also a subspace, as shown in Figure 4.9.

REMARK

Theorem 4.6 states that the *intersection* of two subspaces is a subspace. In Exercise 56 you are asked to show that the *union* of two subspaces is not necessarily a subspace.

THEOREM 4.6 The Intersection of Two Subspaces Is a Subspace

If V and W are both subspaces of a vector space U , then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U .

PROOF

V and W are both subspaces of U , so both contain $\mathbf{0}$, and $V \cap W$ is nonempty. To show that $V \cap W$ is closed under addition, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two vectors in $V \cap W$. V and W are both subspaces of U , which means that both are closed under addition. Both $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V , so their sum $\mathbf{v}_1 + \mathbf{v}_2$ must be in V . Similarly, $\mathbf{v}_1 + \mathbf{v}_2$ is in W because both $\mathbf{v}_1$ and $\mathbf{v}_2$ are also in W . But this implies that $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$, and it follows that $V \cap W$ is closed under addition. It is left to you to show (by a similar argument) that $V \cap W$ is closed under scalar multiplication. (See Exercise 59.)

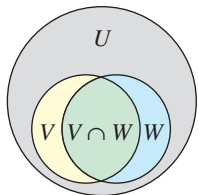


Figure 4.9 The intersection of two subspaces is a subspace.

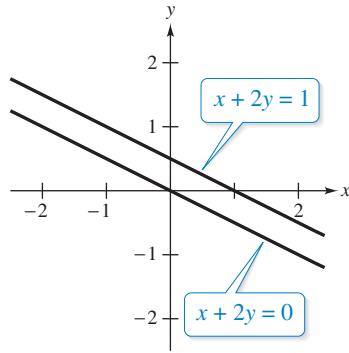


Figure 4.10

SUBSPACES OF R^n

R^n is a convenient source for examples of vector spaces, so the remainder of this section is devoted to looking at subspaces of R^n .

EXAMPLE 6

Determining Subspaces of R^2

Determine whether each subset is a subspace of R^2 .

- The set of points on the line $x + 2y = 0$
- The set of points on the line $x + 2y = 1$

SOLUTION

- Solving for x , a point in R^2 is on the line $x + 2y = 0$ if and only if it has the form $(-2t, t)$, where t is any real number. (See Figure 4.10.)

To show that this set is closed under addition, let $\mathbf{v}_1 = (-2t_1, t_1)$ and $\mathbf{v}_2 = (-2t_2, t_2)$ be any two points on the line. Then you have

$$\mathbf{v}_1 + \mathbf{v}_2 = (-2t_1, t_1) + (-2t_2, t_2) = (-2(t_1 + t_2), t_1 + t_2) = (-2t_3, t_3)$$

where $t_3 = t_1 + t_2$. $\mathbf{v}_1 + \mathbf{v}_2$ lies on the line, and the set is closed under addition. In a similar way, you can show that the set is closed under scalar multiplication. So, this set is a subspace of R^2 .

- This subset of R^2 is *not* a subspace of R^2 because every subspace must contain the zero vector $(0, 0)$, which is not on the line $x + 2y = 1$. (See Figure 4.10.)

Of the two lines in Example 6, the one that is a subspace of R^2 is the one that passes through the origin. This is characteristic of subspaces of R^2 . That is, if W is a subset of R^2 , then it is a subspace if and only if it has one of the forms listed below.

- W consists of the *single point* $(0, 0)$.
- W consists of all points on a *line* that passes through the origin.
- W consists of all of R^2 .

Figure 4.11 shows these three possibilities graphically.

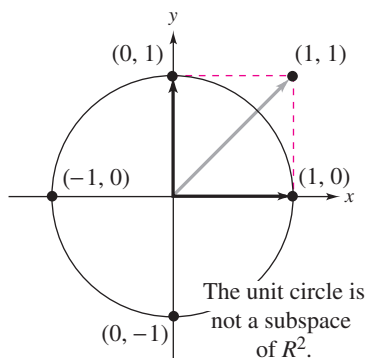


Figure 4.12

REMARK

Another way to tell that the subset shown in Figure 4.12 is not a subspace of R^2 is by noting that it does not contain the zero vector (the origin).

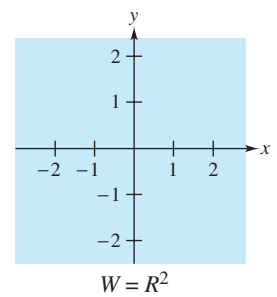
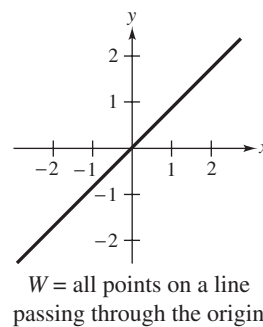
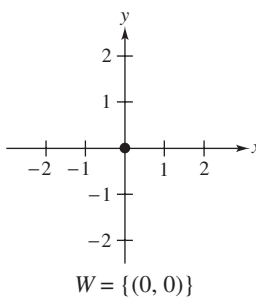


Figure 4.11

EXAMPLE 7

A Subset of R^2 That Is Not a Subspace

Show that the subset of R^2 consisting of all points on $x^2 + y^2 = 1$ is not a subspace.

SOLUTION

This subset of R^2 is *not* a subspace because the points $(1, 0)$ and $(0, 1)$ are in the subset, but their sum $(1, 1)$ is not. (See Figure 4.12.) So, this subset is not closed under addition.

EXAMPLE 8**Determining Subspaces of R^3**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Determine whether each subset is a subspace of R^3 .

- $W = \{(x_1, x_2, 1): x_1 \text{ and } x_2 \text{ are real numbers}\}$
- $W = \{(x_1, x_1 + x_3, x_3): x_1 \text{ and } x_3 \text{ are real numbers}\}$

SOLUTION

- The zero vector $\mathbf{0} = (0, 0, 0)$ is not in W , so W is *not* a subspace of R^3 .
- This set is nonempty because it contains the zero vector $(0, 0, 0)$. Let

$$\mathbf{v} = (v_1, v_1 + v_3, v_3) \quad \text{and} \quad \mathbf{u} = (u_1, u_1 + u_3, u_3)$$

be two vectors in W , and let c be any real number. W is closed under addition because

$$\begin{aligned} \mathbf{v} + \mathbf{u} &= (v_1 + u_1, v_1 + v_3 + u_1 + u_3, v_3 + u_3) \\ &= (v_1 + u_1, (v_1 + u_1) + (v_3 + u_3), v_3 + u_3) \\ &= (x_1, x_1 + x_3, x_3) \end{aligned}$$

where $x_1 = v_1 + u_1$ and $x_3 = v_3 + u_3$, which means that $\mathbf{v} + \mathbf{u}$ is in W . Similarly, W is closed under scalar multiplication because

$$\begin{aligned} c\mathbf{v} &= (cv_1, c(v_1 + v_3), cv_3) \\ &= (cv_1, cv_1 + cv_3, cv_3) \\ &= (x_1, x_1 + x_3, x_3) \end{aligned}$$

where $x_1 = cv_1$ and $x_3 = cv_3$, which means that $c\mathbf{v}$ is in W . So, W is a subspace of R^3 . 

In Example 8, note that the graph of each subset is a plane in R^3 , but the only subset that is a *subspace* is the one represented by a plane that passes through the origin. (See Figure 4.13.) You can show that a subset W of R^3 is a subspace of R^3 if and only if it has one of the forms listed below.

- W consists of the *single point* $(0, 0, 0)$.
- W consists of all points on a *line* that passes through the origin.
- W consists of all points in a *plane* that passes through the origin.
- W consists of all of R^3 .

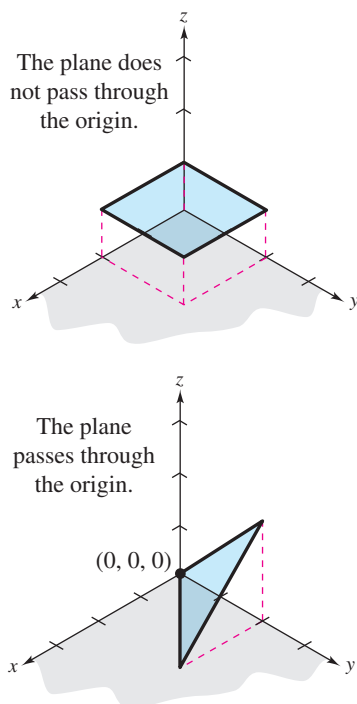
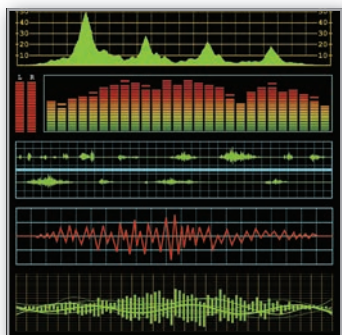


Figure 4.13

**LINEAR ALGEBRA APPLIED**

Digital signal processing depends on sampling, which converts continuous signals into discrete sequences that can be used by digital devices. Traditionally, sampling is uniform and pointwise, and is obtained from a single vector space. Then, the resulting sequence is reconstructed into a continuous-domain signal. Such a process, however, can involve a significant reduction in information, which could result in a low-quality reconstructed signal. In applications such as radar, geophysics, and wireless communications, researchers have determined situations in which sampling from a *union* of vector subspaces can be more appropriate. (Source: *Sampling Signals from a Union of Subspaces—A New Perspective for the Extension of This Theory*, Lu, Y.M. and Do, M.N., *IEEE Signal Processing Magazine*)

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4.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Verifying Subspaces In Exercises 1–6, verify that W is a subspace of V . In each case, assume that V has the standard operations.

1. $W = \{(x_1, x_2, x_3, 0) : x_1, x_2, \text{ and } x_3 \text{ are real numbers}\}$

$$V = \mathbb{R}^4$$

2. $\{(x, y, 4x - 5y) : x \text{ and } y \text{ are real numbers}\}$

$$V = \mathbb{R}^3$$

3. W is the set of all 2×2 matrices of the form

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$

$$V = M_{2,2}$$

4. W is the set of all 3×2 matrices of the form

$$\begin{bmatrix} a & b \\ a - 2b & 0 \\ 0 & c \end{bmatrix}.$$

$$V = M_{3,2}$$

5. **Calculus** W is the set of all functions that are continuous on $[-1, 1]$. V is the set of all functions that are integrable on $[-1, 1]$.

6. **Calculus** W is the set of all functions that are differentiable on $[-1, 1]$. V is the set of all functions that are continuous on $[-1, 1]$.

Subsets That Are Not Subspaces In Exercises 7–20, W is not a subspace of the vector space. Verify this by giving a specific example that violates the test for a vector subspace (Theorem 4.5).

7. W is the set of all vectors in $\mathbb{R}^3$ whose third component is -1 .
8. W is the set of all vectors in $\mathbb{R}^2$ whose first component is 2 .
9. W is the set of all vectors in $\mathbb{R}^2$ whose components are rational numbers.
10. W is the set of all vectors in $\mathbb{R}^2$ whose components are integers.
11. W is the set of all nonnegative functions in $C(-\infty, \infty)$.
12. W is the set of all linear functions $ax + b$, $a \neq 0$, in $C(-\infty, \infty)$.
13. W is the set of all vectors in $\mathbb{R}^3$ whose components are nonnegative.
14. W is the set of all vectors in $\mathbb{R}^3$ whose components are Pythagorean triples.
15. W is the set of all matrices in $M_{3,3}$ of the form

$$\begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 0 \end{bmatrix}.$$

16. W is the set of all matrices in $M_{3,1}$ of the form $[\sqrt{a} \ 0 \ 3a]^T$.

17. W is the set of all matrices in $M_{n,n}$ with determinants equal to 1 .

18. W is the set of all matrices in $M_{n,n}$ such that $A^2 = A$.

19. W is the set of all vectors in $\mathbb{R}^2$ whose second component is the cube of the first.

20. W is the set of all vectors in $\mathbb{R}^2$ whose second component is the square of the first.

Determining Subspaces of $C(-\infty, \infty)$ In Exercises 21–28, determine whether the subset of $C(-\infty, \infty)$ is a subspace of $C(-\infty, \infty)$ with the standard operations. Justify your answer.

21. The set of all positive functions: $f(x) > 0$

22. The set of all negative functions: $f(x) < 0$

23. The set of all even functions: $f(-x) = f(x)$

24. The set of all odd functions: $f(-x) = -f(x)$

25. The set of all constant functions: $f(x) = c$

26. The set of all exponential functions $f(x) = a^x$, where $a > 0$

27. The set of all functions such that $f(0) = 0$

28. The set of all functions such that $f(0) = 1$

Determining Subspaces of $M_{n,n}$ In Exercises 29–36, determine whether the subset of $M_{n,n}$ is a subspace of $M_{n,n}$ with the standard operations. Justify your answer.

29. The set of all $n \times n$ upper triangular matrices

30. The set of all $n \times n$ diagonal matrices

31. The set of all $n \times n$ matrices with integer entries

32. The set of all $n \times n$ matrices A that commute with a given matrix B ; that is, $AB = BA$

33. The set of all $n \times n$ singular matrices

34. The set of all $n \times n$ invertible matrices

35. The set of all $n \times n$ matrices whose entries sum to zero

36. The set of all $n \times n$ matrices whose trace is nonzero
(Recall that the **trace** of a matrix is the sum of the main diagonal entries of the matrix.)

Determining Subspaces of $\mathbb{R}^3$ In Exercises 37–42, determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations. Justify your answer.

37. $W = \{(0, x_2, x_3) : x_2 \text{ and } x_3 \text{ are real numbers}\}$

38. $W = \{(x_1, x_2, 4) : x_1 \text{ and } x_2 \text{ are real numbers}\}$

39. $W = \{(a, a - 3b, b) : a \text{ and } b \text{ are real numbers}\}$

40. $W = \{(s, t, s + t) : s \text{ and } t \text{ are real numbers}\}$

41. $W = \{(x_1, x_2, x_1 x_2) : x_1 \text{ and } x_2 \text{ are real numbers}\}$

42. $W = \{(x_1, 1/x_1, x_3) : x_1 \text{ and } x_3 \text{ are real numbers, } x_1 \neq 0\}$

True or False? In Exercises 43 and 44, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

43. (a) If W is a subspace of a vector space V , then it has closure under scalar multiplication as defined in V .
 (b) If V and W are both subspaces of a vector space U , then the intersection of V and W is also a subspace.
 (c) If U , V , and W are vector spaces such that W is a subspace of V and U is a subspace of V , then $W = U$.
44. (a) Every vector space V contains two proper subspaces that are the zero subspace and itself.
 (b) If W is a subspace of R^2 , then W must contain the vector $(0, 0)$.
 (c) If W is a subspace of a vector space V , then it has closure under addition as defined in V .
 (d) If W is a subspace of a vector space V , then W is also a vector space.

45. Consider the vector spaces

$$P_0, P_1, P_2, \dots, P_n$$

where P_k is the set of all polynomials of degree less than or equal to k , with the standard operations. Show that if $j \leq k$, then P_j is a subspace of P_k .

46. **Calculus** Let W_1, W_2, W_3, W_4 , and W_5 be defined as in Example 5. Show that W_i is a subspace of W_j for $i \leq j$.
47. **Calculus** Let $F(-\infty, \infty)$ be the vector space of real-valued functions defined on the entire real line. Show that each set is a subspace of $F(-\infty, \infty)$.
 (a) $C(-\infty, \infty)$
 (b) The set of all differentiable functions f defined on the real number line
 (c) The set of all differentiable functions f defined on the real number line that satisfy the differential equation $f' - 3f = 0$

48. **Calculus** Determine whether the set

$$S = \left\{ f \in C[0, 1] : \int_0^1 f(x) dx = 0 \right\}$$

is a subspace of $C[0, 1]$. Prove your answer.

49. Let W be the subset of R^3 consisting of all points on a line that passes through the origin. Such a line can be represented by the parametric equations
- $$x = at, \quad y = bt, \quad \text{and} \quad z = ct.$$

Use these equations to show that W is a subspace of R^3 .

50. CAPSTONE Explain why it is sufficient to test for closure to establish that a nonempty subset of a vector space is a subspace.

51. **Guided Proof** Prove that a nonempty set W is a subspace of a vector space V if and only if $ax + by$ is an element of W for all scalars a and b and all vectors x and y in W .

Getting Started: In one direction, assume W is a subspace, and show by using closure axioms that $ax + by$ is an element of W . In the other direction, assume $ax + by$ is an element of W for all scalars a and b and all vectors x and y in W , and verify that W is closed under addition and scalar multiplication.

- (i) If W is a subspace of V , then use scalar multiplication closure to show that ax and by are in W . Now use additive closure to get the desired result.
 (ii) Conversely, assume $ax + by$ is in W . By cleverly assigning specific values to a and b , show that W is closed under addition and scalar multiplication.

52. Let x , y , and z be vectors in a vector space V . Show that the set of all linear combinations of x , y , and z ,

$$W = \{ax + by + cz : a, b, \text{ and } c \text{ are scalars}\}$$

is a subspace of V . This subspace is the **span** of $\{x, y, z\}$.

53. **Proof** Let A be a fixed 2×3 matrix. Prove that the set

$$W = \left\{ x \in R^3 : Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is not a subspace of R^3 .

54. **Proof** Let A be a fixed $m \times n$ matrix. Prove that the set

$$W = \{x \in R^n : Ax = 0\}$$

is a subspace of R^n .

55. **Proof** Let W be a subspace of the vector space V . Prove that the zero vector in V is also the zero vector in W .
56. Give an example showing that the union of two subspaces of a vector space V is not necessarily a subspace of V .
57. **Proof** Let A and B be fixed 2×2 matrices. Prove that the set

$$W = \{X : XAB = BAX\}$$

is a subspace of $M_{2,2}$.

58. **Proof** Let V and W be two subspaces of a vector space U .

- (a) Prove that the set

$$V + W = \{u : u = v + w, v \in V \text{ and } w \in W\}$$

is a subspace of U .

- (b) Describe $V + W$ when V and W are the subspaces of $U = R^2$:

$$V = \{(x, 0) : x \text{ is a real number}\} \text{ and } W = \{(0, y) : y \text{ is a real number}\}.$$

59. **Proof** Complete the proof of Theorem 4.6 by showing that the intersection of two subspaces of a vector space is closed under scalar multiplication.

4.4 Spanning Sets and Linear Independence

-  Write a linear combination of a set of vectors in a vector space V .
-  Determine whether a set S of vectors in a vector space V is a spanning set of V .
-  Determine whether a set of vectors in a vector space V is linearly independent.

LINEAR COMBINATIONS OF VECTORS IN A VECTOR SPACE

This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

Definition of a Linear Combination of Vectors

A vector $\mathbf{v}$ in a vector space V is a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_k\mathbf{u}_k$$

where $c_1, c_2, \dots, c_k$ are scalars.

Often, one or more of the vectors in a set can be written as linear combinations of other vectors in the set. Examples 1 and 2 illustrate this possibility.

EXAMPLE 1

Examples of Linear Combinations

- a. For the set of vectors in $\mathbb{R}^3$

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\}$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 = 3(0, 1, 2) + (1, 0, -5) = (1, 3, 1).$$

- b. For the set of vectors in $M_{2,2}$

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}} \right\}$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4 \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

In Example 1, it is relatively easy to verify that one of the vectors in the set S is a linear combination of the other vectors because the coefficients to form the linear combination are given. Example 2 demonstrates a procedure for finding the coefficients.

EXAMPLE 2**Finding a Linear Combination**

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set

$$S = \{ \overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)} \}.$$

SOLUTION

Find scalars c_1 , c_2 , and c_3 such that

$$\begin{aligned} (1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1) \\ &= (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3). \end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned} c_1 - c_3 &= 1 \\ 2c_1 + c_2 &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}$$

Using Gauss-Jordan elimination, the augmented matrix of this system row reduces to

$$\left[\begin{array}{cccc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

So, this system has infinitely many solutions, each of the form

$$c_1 = 1 + t, \quad c_2 = -1 - 2t, \quad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

(Verify this.) Other choices for t would yield different ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. 

EXAMPLE 3**Finding a Linear Combination**

If possible, write the vector

$$\mathbf{w} = (1, -2, 2)$$

as a linear combination of vectors in the set S in Example 2.

SOLUTION

Following the procedure from Example 2 results in the system

$$\begin{aligned} c_1 - c_3 &= 1 \\ 2c_1 + c_2 &= -2 \\ 3c_1 + 2c_2 + c_3 &= 2. \end{aligned}$$

The augmented matrix of this system row reduces to

$$\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

From the third row you can conclude that the system of equations is inconsistent, which means that there is no solution. Consequently, $\mathbf{w}$ *cannot* be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. 

SPANNING SETS

If every vector in a vector space can be written as a linear combination of vectors in a set S , then S is a **spanning set** of the vector space.

Definition of a Spanning Set of a Vector Space

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a subset of a vector space V . The set S is a **spanning set** of V when *every* vector in V can be written as a linear combination of vectors in S . In such cases it is said that S **spans** V .

EXAMPLE 4

Examples of Spanning Sets

- a. The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

- b. The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

The spanning sets in Example 4 are called the **standard spanning sets** of R^3 and P_2 , respectively. (You will learn more about standard spanning sets in the next section.) In the next example, you will look at a nonstandard spanning set of R^3 .

EXAMPLE 5

A Spanning Set of R^3

Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3 .

SOLUTION

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3 . Find scalars c_1, c_2 , and c_3 such that

$$\begin{aligned} (u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3). \end{aligned}$$

This vector equation produces the system

$$\begin{aligned} c_1 - 2c_3 &= u_1 \\ 2c_1 + c_2 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3. \end{aligned}$$

REMARK

The coefficient matrix of the system in Example 3,

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

has a determinant of zero. (Verify this.)

The coefficient matrix of this system has a nonzero determinant (verify that it is equal to -1), and it follows from the list of equivalent conditions in Section 3.3 that the system has a unique solution. So, any vector in R^3 can be written as a linear combination of the vectors in S , and you can conclude that the set S spans R^3 .

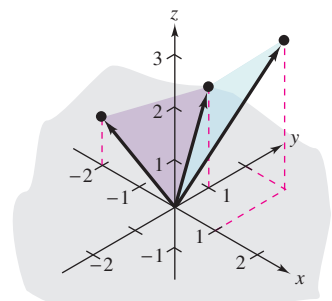
EXAMPLE 6

A Set That Does Not Span R^3

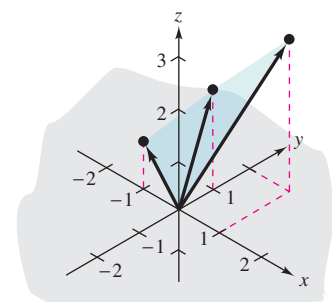
From Example 3 you know that the set

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$$

does not span R^3 because $\mathbf{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S .



$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in S_1 do not lie in a common plane.



$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in S_2 lie in a common plane.

Figure 4.14

Comparing the sets of vectors in Examples 5 and 6, note that the sets are the same except for a seemingly insignificant difference in the third vector.

$$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\} \quad \text{Example 5}$$

$$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\} \quad \text{Example 6}$$

The difference, however, is significant, because the set S_1 spans R^3 whereas the set S_2 does not. The reason for this difference can be seen in Figure 4.14. The vectors in S_2 lie in a common plane; the vectors in S_1 do not.

Although the set S_2 does not span all of R^3 , it does span a subspace of R^3 —namely, the plane in which the three vectors of S_2 lie. This subspace is the **span of S_2** , as stated in the next definition.

Definition of the Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space V , then the **span of S** is the set of all linear combinations of the vectors in S ,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k : c_1, c_2, \dots, c_k \text{ are real numbers}\}.$$

The span of S is denoted by

$$\text{span}(S) \quad \text{or} \quad \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}.$$

When $\text{span}(S) = V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, or that S **spans V** .

The next theorem tells you that the span of any finite nonempty subset of a vector space V is a subspace of V .

THEOREM 4.7 $\text{span}(S)$ Is a Subspace of V

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space V , then $\text{span}(S)$ is a subspace of V . Moreover, $\text{span}(S)$ is the smallest subspace of V that contains S , in the sense that every other subspace of V that contains S must contain $\text{span}(S)$.

PROOF

To show that $\text{span}(S)$, the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, is a subspace of V , show that it is closed under addition and scalar multiplication. Consider any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\text{span}(S)$,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$$

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_k\mathbf{v}_k$$

where

$$c_1, c_2, \dots, c_k \quad \text{and} \quad d_1, d_2, \dots, d_k$$

are scalars. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \dots + (c_k + d_k)\mathbf{v}_k$$

and

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \dots + (cc_k)\mathbf{v}_k$$

which means that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are also in $\text{span}(S)$ because they can be written as linear combinations of vectors in S . So, $\text{span}(S)$ is a subspace of V . It is left to you to prove that $\text{span}(S)$ is the smallest subspace of V that contains S . (See Exercise 59.)

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

For a set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$$

in a vector space V , the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

always has the trivial solution

$$c_1 = 0, c_2 = 0, \dots, c_k = 0.$$

Sometimes, however, there are also nontrivial solutions. For instance, in Example 1(a) you saw that in the set

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\}$$

the vector $\mathbf{v}_1$ can be written as a linear combination of the other two vectors, as shown below.

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3$$

So, the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution in which the coefficients are *not all zero*:

$$c_1 = 1, \quad c_2 = -3, \quad c_3 = -1.$$

When a nontrivial solution exists, the set S is **linearly dependent**. Had the only solution been the trivial one ($c_1 = c_2 = c_3 = 0$), then the set S would have been **linearly independent**. This concept is essential to the study of linear algebra.

Definition of Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is **linearly independent** when the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \dots, c_k = 0.$$

If there are also nontrivial solutions, then S is **linearly dependent**.

EXAMPLE 7

Examples of Linearly Dependent Sets

a. The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

b. The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

c. The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

The next example demonstrates a test to determine whether a set of vectors is linearly independent or linearly dependent.

EXAMPLE 8**Testing for Linear Independence**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Determine whether the set of vectors in R^3 is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

SOLUTION

To test for linear independence or linear dependence, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

If the only solution of this equation is $c_1 = c_2 = c_3 = 0$, then the set S is linearly independent. Otherwise, S is linearly dependent. Expanding this equation, you have

$$c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) = (0, 0, 0)$$

$$(c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) = (0, 0, 0)$$

which yields the homogeneous system of linear equations in c_1 , c_2 , and c_3 below.

$$\begin{aligned} c_1 - 2c_3 &= 0 \\ 2c_1 + c_2 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gauss-Jordan elimination as shown.

$$\left[\begin{array}{cccc} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

This implies that the only solution is the trivial solution $c_1 = c_2 = c_3 = 0$. So, S is linearly independent. 

The steps in Example 8 are summarized below.

Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in a vector space V . To determine whether S is linearly independent or linearly dependent, use the steps below.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$, write a system of linear equations in the variables $c_1, c_2, \dots, c_k$.
2. Determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \dots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

**LINEAR ALGEBRA APPLIED**

Image morphing is the process of transforming one image into another by generating a sequence of synthetic intermediate images. Morphing has a wide variety of applications, such as movie special effects, age progression software, and simulating wound healing and cosmetic surgery results. Morphing an image uses a process called warping, in which a piece of an image is distorted. The mathematics behind warping and morphing can include forming a linear combination of the vectors that bound a triangular piece of an image, and performing an *affine transformation* to form new vectors and a distorted image piece.

dundanim/Shutterstock.com

EXAMPLE 9**Testing for Linear Independence**

Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

SOLUTION

Expanding the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produces

$$\begin{aligned} c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) &= 0 + 0x + 0x^2 \\ (c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 &= 0 + 0x + 0x^2. \end{aligned}$$

Equating corresponding coefficients of powers of x yields the homogeneous system of linear equations in c_1 , c_2 , and c_3 below.

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gaussian elimination as shown below.

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This implies that the system has infinitely many solutions. So, the system must have nontrivial solutions, and you can conclude that the set S is linearly dependent.

One nontrivial solution is

$$c_1 = 2, \quad c_2 = -1, \quad \text{and} \quad c_3 = 3$$

which yields the nontrivial linear combination

$$(2)(1 + x - 2x^2) + (-1)(2 + 5x - x^2) + (3)(x + x^2) = 0.$$

EXAMPLE 10**Testing for Linear Independence**

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

SOLUTION

From the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, you have

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which produces the system of linear equations in c_1 , c_2 , and c_3 below.

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

Use Gaussian elimination to show that the system has only the trivial solution, which means that the set S is linearly independent.

EXAMPLE 11 Testing for Linear Independence

Determine whether the set of vectors in $M_{4,1}$ is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

SOLUTION

From the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$, you obtain

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This equation produces the system of linear equations in c_1, c_2, c_3 , and c_4 below.

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_2 + 3c_3 + c_4 &= 0 \\ -c_1 + c_3 - c_4 &= 0 \\ 2c_2 - 2c_3 + 2c_4 &= 0 \end{aligned}$$

Use Gaussian elimination to show that the system has only the trivial solution, which means that the set S is linearly independent. 

If a set of vectors is linearly dependent, then by definition the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has a nontrivial solution (a solution for which not all the c_i 's are zero). For instance, if $c_1 \neq 0$, then you can solve this equation for $\mathbf{v}_1$ and write $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2, \mathbf{v}_3, \dots$, and $\mathbf{v}_k$. In other words, the vector $\mathbf{v}_1$ *depends* on the other vectors in the set. This property is characteristic of a linearly dependent set.

THEOREM 4.8 A Property of Linearly Dependent Sets

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ can be written as a linear combination of the other vectors in S .

PROOF

To prove the theorem in one direction, assume S is a linearly dependent set. Then there exist scalars $c_1, c_2, c_3, \dots, c_k$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

One of the coefficients must be nonzero, so no generality is lost by assuming $c_1 \neq 0$. Then solving for $\mathbf{v}_1$ as a linear combination of the other vectors produces

$$\begin{aligned} c_1\mathbf{v}_1 &= -c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_k\mathbf{v}_k \\ \mathbf{v}_1 &= -\frac{c_2}{c_1}\mathbf{v}_2 - \frac{c_3}{c_1}\mathbf{v}_3 - \cdots - \frac{c_k}{c_1}\mathbf{v}_k. \end{aligned}$$

Conversely, assume the vector $\mathbf{v}_1$ in S is a linear combination of the other vectors. That is,

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k.$$

Then the equation $-\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has at least one coefficient, -1 , that is nonzero, and you can conclude that S is linearly dependent. 

EXAMPLE 12**Writing a Vector as a Linear Combination of Other Vectors**

In Example 9, you determined that the set

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

is linearly dependent. Show that one of the vectors in this set can be written as a linear combination of the other two.

SOLUTION

In Example 9, the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produced the system

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0. \end{aligned}$$

This system has infinitely many solutions represented by $c_3 = 3t$, $c_2 = -t$, and $c_1 = 2t$. Letting $t = 1$ results in the equation $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$. So, $\mathbf{v}_2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, as shown below.

$$\mathbf{v}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_3$$

A check yields

$$2 + 5x - x^2 = 2(1 + x - 2x^2) + 3(x + x^2) = 2 + 5x - x^2.$$

Theorem 4.8 has a practical corollary that provides a simple test for determining whether *two* vectors are linearly dependent. In Exercise 77 you are asked to prove this corollary.

REMARK

The zero vector is always a scalar multiple of another vector in a vector space.

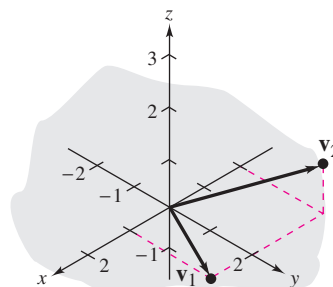
THEOREM 4.8 Corollary

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

EXAMPLE 13**Testing for Linear Dependence of Two Vectors**

- The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 2, 0), (-2, 2, 1)\}$ is linearly independent because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other, as shown in Figure 4.15(a).
- The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(4, -4, -2), (-2, 2, 1)\}$ is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$, as shown in Figure 4.15(b).

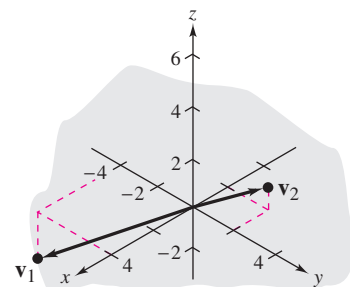
a.



$$S = \{(1, 2, 0), (-2, 2, 1)\}$$

The set S is linearly independent.

b.



$$S = \{(4, -4, -2), (-2, 2, 1)\}$$

The set S is linearly dependent.

Figure 4.15

4.4 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Linear Combinations In Exercises 1–4, write each vector as a linear combination of the vectors in S (if possible).

1. $S = \{(2, -1, 3), (5, 0, 4)\}$
 - (a) $\mathbf{z} = (-1, -2, 2)$
 - (b) $\mathbf{v} = (8, -\frac{1}{4}, \frac{27}{4})$
 - (c) $\mathbf{w} = (1, -8, 12)$
 - (d) $\mathbf{u} = (1, 1, -1)$
2. $S = \{(1, 2, -2), (2, -1, 1)\}$
 - (a) $\mathbf{z} = (-4, -3, 3)$
 - (b) $\mathbf{v} = (-2, -6, 4)$
 - (c) $\mathbf{w} = (-1, -22, 22)$
 - (d) $\mathbf{u} = (1, -5, -5)$
3. $S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$
 - (a) $\mathbf{u} = (-1, 5, -6)$
 - (b) $\mathbf{v} = (-3, 15, 18)$
 - (c) $\mathbf{w} = (\frac{1}{3}, \frac{4}{3}, \frac{1}{2})$
 - (d) $\mathbf{z} = (2, 20, -3)$
4. $S = \{(6, -7, 8, 6), (4, 6, -4, 1)\}$
 - (a) $\mathbf{u} = (2, 19, -16, -4)$
 - (b) $\mathbf{v} = (\frac{49}{2}, \frac{99}{4}, -14, \frac{19}{2})$
 - (c) $\mathbf{w} = (-4, -14, \frac{27}{2}, \frac{53}{8})$
 - (d) $\mathbf{z} = (8, 4, -1, \frac{17}{4})$

Linear Combinations In Exercises 5–8, for the matrices

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$$

in $M_{2,2}$, determine whether the given matrix is a linear combination of A and B .

5. $\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$
6. $\begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$
7. $\begin{bmatrix} -2 & 23 \\ 0 & -9 \end{bmatrix}$
8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Spanning Sets In Exercises 9–18, determine whether the set S spans R^2 . If the set does not span R^2 , then give a geometric description of the subspace that it does span.

9. $S = \{(2, 1), (-1, 2)\}$
10. $S = \{(-1, 1), (3, 1)\}$
11. $S = \{(5, 0), (5, -4)\}$
12. $S = \{(2, 0), (0, 1)\}$
13. $S = \{(-3, 5)\}$
14. $S = \{(1, 1)\}$
15. $S = \{(-1, 2), (2, -4)\}$
16. $S = \{(0, 2), (1, 4)\}$
17. $S = \{(1, 3), (-2, -6), (4, 12)\}$
18. $S = \{(-1, 2), (2, -1), (1, 1)\}$

Spanning Sets In Exercises 19–24, determine whether the set S spans R^3 . If the set does not span R^3 , then give a geometric description of the subspace that it does span.

19. $S = \{(4, 7, 3), (-1, 2, 6), (2, -3, 5)\}$
20. $S = \{(5, 6, 5), (2, 1, -5), (0, -4, 1)\}$
21. $S = \{(-2, 5, 0), (4, 6, 3)\}$
22. $S = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$
23. $S = \{(1, -2, 0), (0, 0, 1), (-1, 2, 0)\}$
24. $S = \{(1, 0, 3), (2, 0, -1), (4, 0, 5), (2, 0, 6)\}$

25. Determine whether the set $S = \{1, x^2, 2 + x^2\}$ spans P_2 .

26. Determine whether the set

$$S = \{-2x + x^2, 8 + x^3, -x^2 + x^3, -4 + x^2\}$$

spans P_3 .

Testing for Linear Independence In Exercises 27–40, determine whether the set S is linearly independent or linearly dependent.

27. $S = \{(-2, 2), (3, 5)\}$
28. $S = \{(3, -6), (-1, 2)\}$
29. $S = \{(0, 0), (1, -1)\}$
30. $S = \{(1, 0), (1, 1), (2, -1)\}$
31. $S = \{(1, -4, 1), (6, 3, 2)\}$
32. $S = \{(6, 2, 1), (-1, 3, 2)\}$
33. $S = \{(-2, 1, 3), (2, 9, -3), (2, 3, -3)\}$
34. $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$
35. $S = \{(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), (3, 4, \frac{7}{2}), (-\frac{3}{2}, 6, 2)\}$
36. $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$
37. $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$
38. $S = \{(4, -3, 6, 2), (1, 8, 3, 1), (3, -2, -1, 0)\}$
39. $S = \{(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$
40. $S = \{(4, 1, 2, 3), (3, 2, 1, 4), (1, 5, 5, 9), (1, 3, 9, 7)\}$

Testing for Linear Independence In Exercises 41–48, determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

41. $S = \{2 - x, 2x - x^2, 6 - 5x + x^2\}$
42. $S = \{-1 + x^2, 5 + 2x\}$
43. $S = \{1 + 3x + x^2, -1 + x + 2x^2, 4x\}$
44. $S = \{x^2, 1 + x^2\}$
45. $S = \{-x + x^2, -5 + x, -5 + x^2\}$
46. $S = \{-2 - x, 2 + 3x + x^2, 6 + 5x + x^2\}$
47. $S = \{7 - 3x + 4x^2, 6 + 2x - x^2, 1 - 8x + 5x^2\}$
48. $S = \{7 - 4x + 4x^2, 6 + 2x - 3x^2, 20 - 6x + 5x^2\}$

Testing for Linear Independence In Exercises 49–52, determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

49. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}$
50. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
51. $A = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & -8 \\ 22 & 23 \end{bmatrix}$
52. $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, B = \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}, C = \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix}$

Showing Linear Dependence In Exercises 53–56, show that the set is linearly dependent by finding a nontrivial linear combination of vectors in the set whose sum is the zero vector. Then express one of the vectors in the set as a linear combination of the other vectors in the set.

53. $S = \{(3, 4), (-1, 1), (2, 0)\}$

54. $S = \{(2, 4), (-1, -2), (0, 6)\}$

55. $S = \{(1, 1, 1), (1, 1, 0), (0, 1, 1), (0, 0, 1)\}$

56. $S = \{(1, 2, 3, 4), (1, 0, 1, 2), (1, 4, 5, 6)\}$

57. For which values of t is each set linearly independent?

(a) $S = \{(t, 1, 1), (1, t, 1), (1, 1, t)\}$

(b) $S = \{(t, 1, 1), (1, 0, 1), (1, 1, 3t)\}$

58. For which values of t is each set linearly independent?

(a) $S = \{(t, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(b) $S = \{(t, t, t), (t, 1, 0), (t, 0, 1)\}$

59. **Proof** Complete the proof of Theorem 4.7.

60. CAPSTONE By inspection, determine why each of the sets is linearly dependent.

(a) $S = \{(1, -2), (2, 3), (-2, 4)\}$

(b) $S = \{(1, -6, 2), (2, -12, 4)\}$

(c) $S = \{(0, 0), (1, 0)\}$

Spanning the Same Subspace In Exercises 61 and 62, show that the sets S_1 and S_2 span the same subspace of R^3 .

61. $S_1 = \{(1, 2, -1), (0, 1, 1), (2, 5, -1)\}$

$S_2 = \{(-2, -6, 0), (1, 1, -2)\}$

62. $S_1 = \{(0, 0, 1), (0, 1, 1), (2, 1, 1)\}$

$S_2 = \{(1, 1, 1), (1, 1, 2), (2, 1, 1)\}$

True or False? In Exercises 63 and 64, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

63. (a) A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space is linearly dependent when the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$ has only the trivial solution.

(b) The set $S = \{(1, 0, 0, 0), (0, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans R^4 .

64. (a) A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors $\mathbf{v}_i$ can be written as a linear combination of the other vectors in S .

(b) If a subset S spans a vector space V , then every vector in V can be written as a linear combination of the vectors in S .

Proof In Exercises 65 and 66, prove that the set of vectors is linearly independent and spans R^3 .

65. $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

66. $B = \{(1, 2, 3), (3, 2, 1), (0, 0, 1)\}$

67. Guided Proof Prove that a nonempty subset of a finite set of linearly independent vectors is linearly independent.

Getting Started: You need to show that a subset of a linearly independent set of vectors cannot be linearly dependent.

(i) Assume S is a set of linearly independent vectors. Let T be a subset of S .

(ii) If T is linearly dependent, then there exist constants not all zero satisfying the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$.

(iii) Use this fact to derive a contradiction and conclude that T is linearly independent.

68. Proof Prove that if S_1 is a nonempty subset of the finite set S_2 , and S_1 is linearly dependent, then so is S_2 .

69. Proof Prove that any set of vectors containing the zero vector is linearly dependent.

70. Proof When the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent and the set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\}$ is linearly dependent, prove that $\mathbf{v}$ is a linear combination of the $\mathbf{u}_i$'s.

71. Proof Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a linearly independent set of vectors in a vector space V . Delete the vector $\mathbf{v}_k$ from this set and prove that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$ cannot span V .

72. Proof When V is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ and one of these vectors can be written as a linear combination of the other $k - 1$ vectors, prove that the span of these $k - 1$ vectors is also V .

73. Proof Let $S = \{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set. Prove that the set $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

74. Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be any three vectors from a vector space V . Determine whether the set of vectors $\{\mathbf{v} - \mathbf{u}, \mathbf{w} - \mathbf{v}, \mathbf{u} - \mathbf{w}\}$ is linearly independent or linearly dependent.

75. Proof Let A be a nonsingular matrix of order 3. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $M_{3,1}$, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ is also linearly independent. Explain, by means of an example, why this is not true when A is singular.

76. Let $f_1(x) = 3x$ and $f_2(x) = |x|$. Graph both functions on the interval $-2 \leq x \leq 2$. Show that these functions are linearly dependent in the vector space $C[0, 1]$, but linearly independent in $C[-1, 1]$.

77. Proof Prove the corollary to Theorem 4.8: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if one is a scalar multiple of the other.

4.5 Basis and Dimension

- Recognize bases in the vector spaces R^n , P_n , and $M_{m,n}$.
- Find the dimension of a vector space.

REMARK

This definition tells you that a basis has two features. A basis S must have *enough vectors* to span V , but *not so many vectors* that one of them could be written as a linear combination of the other vectors in S .

BASIS FOR A VECTOR SPACE

In this section, you will continue your study of spanning sets. In particular, you will look at spanning sets in a vector space that are both linearly independent *and* span the entire space. Such a set forms a **basis** for the vector space. (The plural of *basis* is *bases*.)

Definition of Basis

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V is a **basis** for V when the conditions below are true.

1. S spans V .
2. S is linearly independent.

This definition does not imply that every vector space has a basis consisting of a finite number of vectors. This text, however, restricts the discussion to such bases. Moreover, if a vector space V has a basis with a finite number of vectors, then V is **finite dimensional**. Otherwise, V is **infinite dimensional**. [The vector space P of all polynomials is infinite dimensional, as is the vector space $C(-\infty, \infty)$ of all continuous functions defined on the real line.] The vector space $V = \{\mathbf{0}\}$, consisting of the zero vector alone, is finite dimensional.

EXAMPLE 1 The Standard Basis for R^3

Show that the set below is a basis for R^3 .

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

SOLUTION

Example 4(a) in Section 4.4 showed that S spans R^3 . Furthermore, S is linearly independent because the vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

(Verify this.) So, S is a basis for R^3 . (See Figure 4.16.)

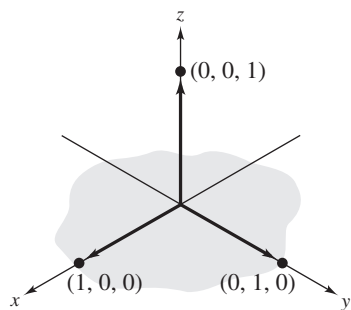


Figure 4.16

The basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

is the **standard basis** for R^3 . This can be generalized to n -space. That is, the vectors

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \dots, 1) \end{aligned}$$

form the **standard basis** for R^n .

The next two examples describe nonstandard bases for R^2 and R^3 .

EXAMPLE 2

A Nonstandard Basis for R^2

Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for R^2 .

SOLUTION

According to the definition of a basis for a vector space, you must show that S spans R^2 and S is linearly independent.

To verify that S spans R^2 , let

$$\mathbf{x} = (x_1, x_2)$$

represent an arbitrary vector in R^2 . To show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, consider the equation

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{x} \\ c_1(1, 1) + c_2(1, -1) &= (x_1, x_2) \\ (c_1 + c_2, c_1 - c_2) &= (x_1, x_2). \end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}$$

The coefficient matrix of this system has a nonzero determinant, which means that the system has a unique solution. So, S spans R^2 .

One way to show that S is linearly independent is to let $(x_1, x_2) = (0, 0)$ in the above system, yielding the homogeneous system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

This system has only the trivial solution

$$c_1 = c_2 = 0.$$

So, S is linearly independent. An alternative way to show that S is linearly independent is to note that

$$\mathbf{v}_1 = (1, 1) \quad \text{and} \quad \mathbf{v}_2 = (1, -1)$$

are not scalar multiples of each other. This means, by the corollary to Theorem 4.8, that $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

You can conclude that S is a basis for R^2 because it is a spanning set for R^2 and it is linearly independent. 

EXAMPLE 3

A Nonstandard Basis for R^3

See LarsonLinearAlgebra.com for an interactive version of this type of example.

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans R^3 and is linearly independent. So, S is a basis for R^3 . 

EXAMPLE 4**A Basis for Polynomials**

Show that the vector space P_3 has the basis

$$S = \{1, x, x^2, x^3\}.$$

SOLUTION

It is clear that S spans P_3 because the span of S consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3, \quad a_0, a_1, a_2, \text{ and } a_3 \text{ are real numbers}$$

which is precisely the form of all polynomials in P_3 .

To verify the linear independence of S , recall that the zero vector $\mathbf{0}$ in P_3 is the polynomial $\mathbf{0}(x) = 0$ for all x . The test for linear independence yields the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \mathbf{0}(x) = 0, \quad \text{for all } x.$$

This third-degree polynomial is *identically equal to zero*. From algebra you know that for a polynomial to be identically equal to zero, all of its coefficients must be zero; that is,

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So, S is linearly independent and is a basis for P_3 . 

REMARK

The basis $S = \{1, x, x^2, x^3\}$ is the **standard basis** for P_3 . Similarly, the **standard basis** for P_n is

$$S = \{1, x, x^2, \dots, x^n\}.$$

EXAMPLE 5**A Basis for $M_{2,2}$**

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

is a basis for $M_{2,2}$. This set is the **standard basis** for $M_{2,2}$. In a similar manner, the standard basis for the vector space $M_{m,n}$ consists of the mn distinct $m \times n$ matrices having a single entry equal to 1 and all the other entries equal to 0. 

THEOREM 4.9 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then every vector in V can be written in one and only one way as a linear combination of vectors in S .

PROOF

The existence portion of the proof is straightforward. That is, S spans V , so you know that an arbitrary vector $\mathbf{u}$ in V can be expressed as $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$.

To prove uniqueness (that a vector can be represented in only one way), assume $\mathbf{u}$ has another representation

$$\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n.$$

Subtracting the second representation from the first produces

$$\mathbf{u} - \mathbf{u} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \dots + (c_n - b_n)\mathbf{v}_n = \mathbf{0}.$$

S is linearly independent, however, so the only solution to this equation is the trivial solution

$$c_1 - b_1 = 0, \quad c_2 - b_2 = 0, \quad \dots, \quad c_n - b_n = 0$$

which means that $c_i = b_i$ for all $i = 1, 2, \dots, n$, and $\mathbf{u}$ has only one representation for the basis S . 

EXAMPLE 6**Uniqueness of Basis Representation**

Let $\mathbf{u} = \{u_1, u_2, u_3\}$ be any vector in R^3 . Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

SOLUTION

From the equation

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)\end{aligned}$$

you obtain the system of linear equations below.

$$\begin{array}{rcl} c_1 & - & 2c_3 = u_1 \\ 2c_1 + & c_2 & = u_2 \\ 3c_1 + 2c_2 + & c_3 & = u_3 \end{array} \quad \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\mathbf{A} \qquad \mathbf{c} \qquad \mathbf{u}$

The matrix A is invertible, so you know this system has a unique solution, $\mathbf{c} = A^{-1}\mathbf{u}$. Verify by finding A^{-1} that

$$\begin{aligned}c_1 &= -u_1 + 4u_2 - 2u_3 \\ c_2 &= 2u_1 - 7u_2 + 4u_3 \\ c_3 &= -u_1 + 2u_2 - u_3.\end{aligned}$$

For example, $\mathbf{u} = (1, 0, 0)$ can be represented uniquely as $-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$. 

You will now study two important theorems concerning bases.

THEOREM 4.10 Bases and Linear Dependence

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then every set containing more than n vectors in V is linearly dependent.

PROOF

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be any set of m vectors in V , where $m > n$. To show that S_1 is linearly dependent, you need to find scalars $k_1, k_2, \dots, k_m$ (not all zero) such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \dots + k_m\mathbf{u}_m = \mathbf{0}. \quad \text{Equation 1}$$

S is a basis for V , so each $\mathbf{u}_i$ can be represented as a linear combination of vectors in S :

$$\begin{aligned}\mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \dots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \dots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \dots + c_{nm}\mathbf{v}_n.\end{aligned}$$

Substituting into Equation 1 and regrouping terms produces

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n = \mathbf{0}$$

where $d_i = c_{i1}k_1 + c_{i2}k_2 + \dots + c_{im}k_m$. The $\mathbf{v}_i$'s form a linearly independent set, so each $d_i = 0$, and you obtain the system of equations below.

$$\begin{aligned}c_{11}k_1 + c_{12}k_2 + \dots + c_{1m}k_m &= 0 \\ c_{21}k_1 + c_{22}k_2 + \dots + c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1 + c_{n2}k_2 + \dots + c_{nm}k_m &= 0\end{aligned}$$

But this homogeneous system has fewer equations than variables $k_1, k_2, \dots, k_m$, and from Theorem 1.1, it has *nontrivial* solutions. Consequently, S_1 is linearly dependent. 

EXAMPLE 7**Linearly Dependent Sets in R^3 and P_3**

- a. R^3 has a basis consisting of three vectors, so the set

$$S = \{(1, 2, -1), (1, 1, 0), (2, 3, 0), (5, 9, -1)\}$$

must be linearly dependent.

- b. P_3 has a basis consisting of four vectors, so the set

$$S = \{1, 1 + x, 1 - x, 1 + x + x^2, 1 - x + x^2\}$$

must be linearly dependent. 

R^n has the standard basis consisting of n vectors, so it follows from Theorem 4.10 that every set of vectors in R^n containing more than n vectors must be linearly dependent. The next theorem states another significant consequence of Theorem 4.10.

THEOREM 4.11 Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

PROOF

Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V , and let $S_2 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be any other basis for V . Theorem 4.10 implies that $m \leq n$, because S_1 is a basis and S_2 is linearly independent. Similarly, $n \leq m$ because S_1 is linearly independent and S_2 is a basis. Consequently, $n = m$. 

EXAMPLE 8**Spanning Sets and Bases**

Use Theorem 4.11 to explain why each statement is true.

- a. The set $S_1 = \{(3, 2, 1), (7, -1, 4)\}$ is not a basis for R^3 .
 b. The set $S_2 = \{2 + x, x^2, -1 + x^3, 1 + 3x, 3 - 2x + x^2\}$ is not a basis for P_3 .

SOLUTION

- a. The standard basis for R^3 , $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, has three vectors, and S_1 has only two vectors. By Theorem 4.11, S_1 cannot be a basis for R^3 .
 b. The standard basis for P_3 , $S = \{1, x, x^2, x^3\}$, has four vectors. By Theorem 4.11, the set S_2 has too many vectors to be a basis for P_3 . 

**LINEAR ALGEBRA APPLIED**

The RGB color model uses combinations of red (**r**), green (**g**), and blue (**b**), known as the *primary additive colors*, to create all other colors in a system. Using the standard basis for R^3 , where $\mathbf{r} = (1, 0, 0)$, $\mathbf{g} = (0, 1, 0)$, and $\mathbf{b} = (0, 0, 1)$, any visible color can be represented as a linear combination $c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b}$ of the primary additive colors. The coefficients c_i are values between 0 and a specified maximum a , inclusive. When $c_1 = c_2 = c_3$, the color is *grayscale*, with $c_i = 0$ representing black and $c_i = a$ representing white. The RGB color model is commonly used in computers, smart phones, televisions, and other electronics with a color display.

Chernetskiy/Shutterstock.com

THE DIMENSION OF A VECTOR SPACE

By Theorem 4.11, if a vector space V has a basis consisting of n vectors, then every other basis for the space also has n vectors. This number n is the **dimension** of V .

Definition of the Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then the number n is the **dimension** of V , denoted by $\dim(V) = n$. When V consists of the zero vector alone, the dimension of V is defined as zero.

This definition allows you to state the dimensions of familiar vector spaces. In each example listed below, the dimension is simply the number of vectors in the standard basis.

1. The dimension of R^n with the standard operations is n .
2. The dimension of P_n with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is mn .

If W is a subspace of a vector space V that has dimension n , then it can be shown that the dimension of W is less than or equal to n . (See Exercise 83.) The next three examples show a technique for finding the dimension of a subspace. Basically, you determine the dimension by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace, and the dimension of the subspace is the number of vectors in the basis.

EXAMPLE 9

Finding Dimensions of Subspaces

Find the dimension of each subspace of R^3 .

- a. $W = \{(d, c - d, c) : c \text{ and } d \text{ are real numbers}\}$
- b. $W = \{(2b, b, 0) : b \text{ is a real number}\}$

SOLUTION

- a. By writing the representative vector $(d, c - d, c)$ as

$$(d, c - d, c) = (0, c, c) + (d, -d, 0) = c(0, 1, 1) + d(1, -1, 0)$$

you can see that W is spanned by the set $S = \{(0, 1, 1), (1, -1, 0)\}$. Using the techniques described in the preceding section, you can show that this set is linearly independent. So, S is a basis for W , and W is a two-dimensional subspace of R^3 .

- b. By writing the representative vector $(2b, b, 0)$ as $b(2, 1, 0)$, you can see that W is spanned by the set $S = \{(2, 1, 0)\}$. So, W is a one-dimensional subspace of R^3 .

REMARK

In Example 9(a), the subspace W is the plane in R^3 determined by the vectors $(0, 1, 1)$ and $(1, -1, 0)$. In Example 9(b), the subspace is the line determined by the vector $(2, 1, 0)$.

EXAMPLE 10

Finding the Dimension of a Subspace

Find the dimension of the subspace W of R^4 spanned by

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}.$$

SOLUTION

Although W is spanned by the set S , S is not a basis for W because S is a linearly dependent set. Specifically, $\mathbf{v}_3$ can be written as $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$. This means that W is spanned by the set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Moreover, S_1 is linearly independent because neither vector is a scalar multiple of the other, and you can conclude that the dimension of W is 2.

EXAMPLE 11**Finding the Dimension of a Subspace**

Let W be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of W ?

SOLUTION

Every 2×2 symmetric matrix has the form

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans W . Moreover, S can be shown to be linearly independent, and you can conclude that the dimension of W is 3. 

Usually, to conclude that a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , you must show that S satisfies two conditions: S spans V and is linearly independent. If V is known to have a dimension of n , however, then the next theorem tells you that you do not need to check both conditions. Either one will suffice. The proof is left as an exercise. (See Exercise 82.)

THEOREM 4.12 Basis Tests in an n -Dimensional Space

Let V be a vector space of dimension n .

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V , then S is a basis for V .
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V , then S is a basis for V .

EXAMPLE 12**Testing for a Basis in an n -Dimensional Space**

Show that the set of vectors is a basis for $M_{5,1}$.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}}, \overset{\mathbf{v}_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}} \right\}$$

SOLUTION

S has five vectors and the dimension of $M_{5,1}$ is 5, so apply Theorem 4.12 to verify that S is a basis by showing either that S is linearly independent or that S spans $M_{5,1}$. To show that S is linearly independent, form the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0}$, which yields the linear system below.

$$\begin{array}{rcl} c_1 & & = 0 \\ 2c_1 + c_2 & & = 0 \\ -c_1 + 3c_2 + 2c_3 & & = 0 \\ 3c_1 - 2c_2 - c_3 + 2c_4 & & = 0 \\ 4c_1 + 3c_2 + 5c_3 - 3c_4 - 2c_5 & = & 0 \end{array}$$

This system has only the trivial solution, so S is linearly independent. By Theorem 4.12, S is a basis for $M_{5,1}$. 

4.5 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Writing the Standard Basis In Exercises 1–6, write the standard basis for the vector space.

1. R^6
2. R^4
3. $M_{3,3}$
4. $M_{4,1}$
5. P_4
6. P_2

Explaining Why a Set Is Not a Basis In Exercises 7–14, explain why S is not a basis for R^2 .

7. $S = \{(-4, 5), (0, 0)\}$
8. $S = \{(2, 3), (6, 9)\}$
9. $S = \{(-3, 2)\}$
10. $S = \{(5, -7)\}$
11. $S = \{(1, 2), (1, 0), (0, 1)\}$
12. $S = \{(-1, 2), (1, -2), (2, 4)\}$
13. $S = \{(6, -5), (12, -10)\}$
14. $S = \{(4, -3), (8, -6)\}$

Explaining Why a Set Is Not a Basis In Exercises 15–22, explain why S is not a basis for R^3 .

15. $S = \{(1, 3, 0), (4, 1, 2), (-2, 5, -2)\}$
16. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$
17. $S = \{(7, 0, 3), (8, -4, 1)\}$
18. $S = \{(1, 1, 2), (0, 2, 1)\}$
19. $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$
20. $S = \{(-1, 0, 0), (0, 0, 1), (1, 0, 0)\}$
21. $S = \{(1, 1, 1), (0, 1, 1), (1, 0, 1), (0, 0, 0)\}$
22. $S = \{(6, 4, 1), (3, -5, 1), (8, 13, 6), (0, 6, 9)\}$

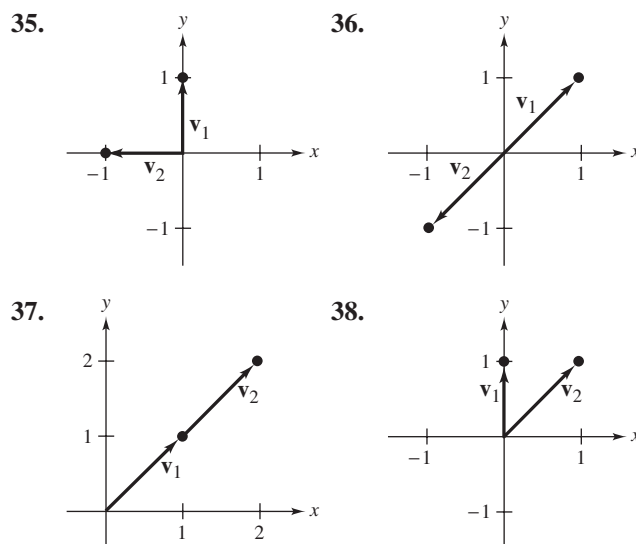
Explaining Why a Set Is Not a Basis In Exercises 23–30, explain why S is not a basis for P_2 .

23. $S = \{1, 2x, -4 + x^2, 5x\}$
24. $S = \{2, x, 3 + x, 3x^2\}$
25. $S = \{-x, 4x^2\}$
26. $S = \{-1, 11x\}$
27. $S = \{1 + x^2, 1 - x^2\}$
28. $S = \{1 - 2x + x^2, 3 - 6x + 3x^2, -2 + 4x - 2x^2\}$
29. $S = \{1 - x, 1 - x^2, -1 - 2x + 3x^2\}$
30. $S = \{-3 + 6x, 3x^2, 1 - 2x - x^2\}$

Explaining Why a Set Is Not a Basis In Exercises 31–34, explain why S is not a basis for $M_{2,2}$.

31. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$
32. $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
33. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} \right\}$
34. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Determining Whether a Set Is a Basis In Exercises 35–38, determine whether the set $\{v_1, v_2\}$ is a basis for R^2 .



Determining Whether a Set Is a Basis In Exercises 39–46, determine whether S is a basis for the given vector space.

39. $S = \{(4, -3), (5, 2)\}$ for R^2
40. $S = \{(1, 2), (1, -1)\}$ for R^2
41. $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for R^3
42. $S = \{(2, 1, 0), (0, -1, 1)\}$ for R^3
43. $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for R^3
44. $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for R^3
45. $S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\}$ for R^4
46. $S = \{(1, 0, 0, 1), (0, 2, 0, 2), (1, 0, 1, 0), (0, 2, 2, 0)\}$ for R^4

Determining Whether a Set Is a Basis In Exercises 47–50, determine whether S is a basis for P_3 .

47. $S = \{1 - 2t^2 + t^3, -4 + t^2, 2t + t^3, 5t\}$
48. $S = \{4t - t^2, 5 + t^3, 5 + 3t, -3t^2 + 2t^3\}$
49. $S = \{4 - t, t^3, 6t^2, 3t + t^3, -1 + 4t\}$
50. $S = \{-1 + t^3, 2t^2, 3 + t, 5 + 2t + 2t^2 + t^3\}$

Determining Whether a Set Is a Basis In Exercises 51 and 52, determine whether S is a basis for $M_{2,2}$.

51. $S = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$
52. $S = \left\{ \begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix} \right\}$

Determining Whether a Set Is a Basis In Exercises 53–56, determine whether S is a basis for R^3 . If it is, write $\mathbf{u} = (8, 3, 8)$ as a linear combination of the vectors in S .

53. $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$

54. $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

55. $S = \{(0, 0, 0), (1, 3, 4), (6, 1, -2)\}$

56. $S = \left\{\left(\frac{2}{3}, \frac{5}{2}, 1\right), \left(1, \frac{3}{2}, 0\right), (2, 12, 6)\right\}$

Finding the Dimension of a Vector Space In Exercises 57–64, find the dimension of the vector space.

57. R^6

58. R

59. P_7

60. P_4

61. $M_{2,3}$

62. $M_{3,2}$

63. R^{3m}

64. $P_{2m-1}, m \geq 1$

65. Find a basis for the vector space of all 3×3 diagonal matrices. What is the dimension of this vector space?

66. Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this vector space?

67. Find all subsets of the set

$$S = \{(1, 0), (0, 1), (1, 1)\}$$

that form a basis for R^2 .

68. Find all subsets of the set

$$S = \{(1, 3, -2), (-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$$

that form a basis for R^3 .

69. Find a basis for R^2 that includes the vector $(2, 2)$.

70. Find a basis for R^3 that includes the vectors $(1, 0, 2)$ and $(0, 1, 1)$.

Geometric Description, Basis, and Dimension In Exercises 71 and 72, (a) give a geometric description of, (b) find a basis for, and (c) find the dimension of the subspace W of R^2 .

71. $W = \{(2t, t) : t \text{ is a real number}\}$

72. $W = \{(0, t) : t \text{ is a real number}\}$

Geometric Description, Basis, and Dimension In Exercises 73 and 74, (a) give a geometric description of, (b) find a basis for, and (c) find the dimension of the subspace W of R^3 .

73. $W = \{(2t, t, -t) : t \text{ is a real number}\}$

74. $W = \{(2s - t, s, t) : s \text{ and } t \text{ are real numbers}\}$

Basis and Dimension In Exercises 75–78, find (a) a basis for and (b) the dimension of the subspace W of R^4 .

75. $W = \{(2s - t, s, t, s) : s \text{ and } t \text{ are real numbers}\}$

76. $W = \{(5t, -3t, t, t) : t \text{ is a real number}\}$

77. $W = \{(0, 6t, t, -t) : t \text{ is a real number}\}$

78. $W = \{(s + 4t, t, s, 2s - t) : s \text{ and } t \text{ are real numbers}\}$

True or False? In Exercises 79 and 80, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

79. (a) If $\dim(V) = n$, then there exists a set of $n - 1$ vectors in V that span V .

(b) If $\dim(V) = n$, then there exists a set of $n + 1$ vectors in V that span V .

80. (a) If $\dim(V) = n$, then any set of $n + 1$ vectors in V must be linearly dependent.

(b) If $\dim(V) = n$, then any set of $n - 1$ vectors in V must be linearly independent.

81. **Proof** Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V and c is a nonzero scalar, then the set $S_1 = \{c\mathbf{v}_1, c\mathbf{v}_2, \dots, c\mathbf{v}_n\}$ is also a basis for V .

82. **Proof** Prove Theorem 4.12.

83. **Proof** Prove that if W is a subspace of a finite dimensional vector space V , then $\dim(W) \leq \dim(V)$.

84. CAPSTONE

(a) A set S_1 consists of two vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Explain why S_1 is not a basis for R^3 .

(b) A set S_2 consists of four vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Explain why S_2 is not a basis for R^3 .

(c) A set S_3 consists of three vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Determine the conditions under which S_3 is a basis for R^3 .

85. **Proof** Let S be a linearly independent set of vectors from a finite dimensional vector space V . Prove that there exists a basis for V containing S .

86. **Guided Proof** Let S be a spanning set for a finite dimensional vector space V . Prove that there exists a subset S' of S that forms a basis for V .

Getting Started: S is a spanning set, but it may not be a basis because it may be linearly dependent. You need to remove extra vectors so that a subset S' is a spanning set and is also linearly independent.

(i) If S is a linearly independent set, then you are done. If not, remove some vector $\mathbf{v}$ from S that is a linear combination of the other vectors in S . Call this set S_1 .

(ii) If S_1 is a linearly independent set, then you are done. If not, then continue to remove dependent vectors until you produce a linearly independent subset S' .

(iii) Conclude that this subset is the minimal spanning set S' .

4.6 Rank of a Matrix and Systems of Linear Equations

-  Find a basis for the row space, a basis for the column space, and the rank of a matrix.
-  Find the nullspace of a matrix.
-  Find the solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$.

ROW SPACE, COLUMN SPACE, AND RANK OF A MATRIX

In this section, you will investigate the vector space spanned by the row vectors (or column vectors) of a matrix. Then you will see how such vector spaces relate to solutions of systems of linear equations.

For an $m \times n$ matrix A , recall that the n -tuples corresponding to the rows of A are the row vectors of A .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{array}{l} \text{Row Vectors of } A \\ (a_{11}, a_{12}, \dots, a_{1n}) \\ (a_{21}, a_{22}, \dots, a_{2n}) \\ \vdots \\ (a_{m1}, a_{m2}, \dots, a_{mn}) \end{array}$$

Similarly, the $m \times 1$ matrices corresponding to the columns of A are the column vectors of A .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{array}{l} \text{Column Vectors of } A \\ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{array}$$

EXAMPLE 1

Row Vectors and Column Vectors

For the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & 4 \end{bmatrix}$, the row vectors are $(0, 1, -1)$ and $(-2, 3, 4)$

and the column vectors are $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

In Example 1, note that for an $m \times n$ matrix A , the row vectors are vectors in R^n and the column vectors are vectors in R^m . This leads to the definitions of the **row space** and **column space** of a matrix listed below.

Definitions of Row Space and Column Space of a Matrix

Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace of R^n spanned by the row vectors of A .
2. The **column space** of A is the subspace of R^m spanned by the column vectors of A .

Recall that two matrices are row-equivalent when one can be obtained from the other by elementary row operations. The next theorem tells you that row-equivalent matrices have the same row space.

REMARK

Theorem 4.13 states that elementary row operations do not change the row space of a matrix. Elementary row operations can, however, change the *column* space of a matrix.

THEOREM 4.13 Row-Equivalent Matrices Have the Same Row Space

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B , then the row space of A is equal to the row space of B .

PROOF

The rows of B can be obtained from the rows of A by elementary row operations (scalar multiplication and addition), so it follows that the row vectors of B can be written as linear combinations of the row vectors of A . The row vectors of B lie in the row space of A , and the subspace spanned by the row vectors of B is contained in the row space of A . But it is also true that the rows of A can be obtained from the rows of B by elementary row operations. So, the two row spaces are subspaces of each other, making them equal. 

If a matrix B is in row-echelon form, then its nonzero row vectors form a linearly independent set. (Verify this.) Consequently, they form a basis for the row space of B , and by Theorem 4.13 they also form a basis for the row space of A . The next theorem states this important result.

THEOREM 4.14 Basis for the Row Space of a Matrix

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A .

EXAMPLE 2**Finding a Basis for a Row Space**

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

SOLUTION

Using elementary row operations, rewrite A in row-echelon form as shown below.

$$B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \end{matrix}$$

By Theorem 4.14, the nonzero row vectors of B , $\mathbf{w}_1 = (1, 3, 1, 3)$, $\mathbf{w}_2 = (0, 1, 1, 0)$, and $\mathbf{w}_3 = (0, 0, 0, 1)$, form a basis for the row space of A . 

The technique used in Example 2 to find a basis for the row space of a matrix can be used to find a basis for the subspace spanned by the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in R^n . Use the vectors in S to form the rows of a matrix A , then use elementary row operations to rewrite A in row-echelon form. The nonzero rows of this matrix will then form a basis for the subspace spanned by S . Example 3 demonstrates this process.

EXAMPLE 3**Finding a Basis for a Subspace**

Find a basis for the subspace of R^3 spanned by

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}.$$

SOLUTION

Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of a matrix A . Then write A in row-echelon form.

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \begin{matrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{matrix} \rightarrow B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{matrix}$$

The nonzero row vectors of B , $\mathbf{w}_1 = (1, -2, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$, form a basis for the row space of A . That is, they form a basis for the subspace spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

To find a basis for the column space of a matrix A , you have two options. On the one hand, you could use the fact that the column space of A is equal to the row space of A^T and apply the technique of Example 2 to the matrix A^T . On the other hand, observe that although row operations can change the column space of a matrix, they do not change the dependency relationships among columns. (You are asked to prove this in Exercise 80.) For example, consider the row-equivalent matrices A and B from Example 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \begin{matrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{matrix} \quad B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{matrix}$$

Notice that columns 1, 2, and 3 of matrix B satisfy $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2$, and the corresponding columns of matrix A satisfy $\mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$. Similarly, the column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_4$ of matrix B are linearly independent, as are the corresponding columns of matrix A .

The next two examples show how to find a basis for the column space of a matrix using these methods.

EXAMPLE 4**Finding a Basis for the Column Space of a Matrix (Method 1)**

Find a basis for the column space of matrix A from Example 2 by finding a basis for the row space of A^T .

SOLUTION

Write the transpose of A and use elementary row operations to write A^T in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \mathbf{w}_4 \end{matrix}$$

So, $\mathbf{w}_1 = (1, 0, -3, 3, 2)$, $\mathbf{w}_2 = (0, 1, 9, -5, -6)$, and $\mathbf{w}_3 = (0, 0, 1, -1, -1)$ form a basis for the row space of A^T . This is equivalent to saying that the column vectors $[1 \ 0 \ -3 \ 3 \ 2]^T$, $[0 \ 1 \ 9 \ -5 \ -6]^T$, and $[0 \ 0 \ 1 \ -1 \ -1]^T$ form a basis for the column space of A .

EXAMPLE 5**Finding a Basis for the Column Space of a Matrix (Method 2)**

Find a basis for the column space of matrix A from Example 2 by using the dependency relationships among columns.

SOLUTION

In Example 2, row operations were used on the original matrix A to obtain its row-echelon form B . As mentioned earlier, in matrix B , the first, second, and fourth column vectors are linearly independent (these columns have the leading 1's), as are the corresponding columns of matrix A . So, a basis for the column space of A consists of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}.$$

REMARK

Notice that the row-echelon form B tells you which columns of A form the basis for the column space. You do not use the column vectors of B to form the basis.

Notice that the basis for the column space obtained in Example 5 is different than that obtained in Example 4. Verify that these bases both span the column space of A by writing the columns of A as linear combinations of the vectors in each basis.

Also notice in Examples 2, 4, and 5 that both the row space and the column space of A have a dimension of 3 (because there are *three* vectors in both bases). The next theorem generalizes this.

THEOREM 4.15 Row and Column Spaces Have Equal Dimensions

The row space and column space of an $m \times n$ matrix A have the same dimension.

PROOF

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be the row vectors and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be the column vectors of

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Assume the row space of A has dimension r and basis $S = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r\}$, where $\mathbf{b}_i = (b_{i1}, b_{i2}, \dots, b_{in})$. Using this basis, write the row vectors of A as

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 + \cdots + c_{1r}\mathbf{b}_r \\ \mathbf{v}_2 &= c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2 + \cdots + c_{2r}\mathbf{b}_r \\ &\vdots \\ \mathbf{v}_m &= c_{m1}\mathbf{b}_1 + c_{m2}\mathbf{b}_2 + \cdots + c_{mr}\mathbf{b}_r. \end{aligned}$$

Rewrite this system of vector equations as shown below.

$$\begin{aligned} (a_{11}, a_{12}, \dots, a_{1n}) &= c_{11}(b_{11}, b_{12}, \dots, b_{1n}) + c_{12}(b_{21}, b_{22}, \dots, b_{2n}) + \cdots + c_{1r}(b_{r1}, b_{r2}, \dots, b_{rn}) \\ (a_{21}, a_{22}, \dots, a_{2n}) &= c_{21}(b_{11}, b_{12}, \dots, b_{1n}) + c_{22}(b_{21}, b_{22}, \dots, b_{2n}) + \cdots + c_{2r}(b_{r1}, b_{r2}, \dots, b_{rn}) \\ &\vdots \\ (a_{m1}, a_{m2}, \dots, a_{mn}) &= c_{m1}(b_{11}, b_{12}, \dots, b_{1n}) + c_{m2}(b_{21}, b_{22}, \dots, b_{2n}) + \cdots + c_{mr}(b_{r1}, b_{r2}, \dots, b_{rn}) \end{aligned}$$

Now, take only entries corresponding to the first column of matrix A to obtain the system of scalar equations shown on the next page.

$$\begin{aligned}
 a_{11} &= c_{11}b_{11} + c_{12}b_{21} + \cdots + c_{1r}b_{r1} \\
 a_{21} &= c_{21}b_{11} + c_{22}b_{21} + \cdots + c_{2r}b_{r1} \\
 &\vdots \\
 a_{m1} &= c_{m1}b_{11} + c_{m2}b_{21} + \cdots + c_{mr}b_{r1}
 \end{aligned}$$

Similarly, for the entries of the j th column, you can obtain the system below.

$$\begin{aligned}
 a_{1j} &= c_{11}b_{1j} + c_{12}b_{2j} + \cdots + c_{1r}b_{rj} \\
 a_{2j} &= c_{21}b_{1j} + c_{22}b_{2j} + \cdots + c_{2r}b_{rj} \\
 &\vdots \\
 a_{mj} &= c_{m1}b_{1j} + c_{m2}b_{2j} + \cdots + c_{mr}b_{rj}
 \end{aligned}$$

Now, let the vectors

$$\mathbf{c}_i = [c_{1i} \ c_{2i} \ \cdots \ c_{mi}]^T.$$

Then the system for the j th column can be rewritten in a vector form as

$$\mathbf{u}_j = b_{1j}\mathbf{c}_1 + b_{2j}\mathbf{c}_2 + \cdots + b_{rj}\mathbf{c}_r.$$

Put all column vectors together to obtain

$$\begin{aligned}
 \mathbf{u}_1 &= [a_{11} \ a_{12} \ \cdots \ a_{m1}]^T = b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + \cdots + b_{r1}\mathbf{c}_r \\
 \mathbf{u}_2 &= [a_{12} \ a_{22} \ \cdots \ a_{m2}]^T = b_{12}\mathbf{c}_1 + b_{22}\mathbf{c}_2 + \cdots + b_{r2}\mathbf{c}_r \\
 &\vdots \\
 \mathbf{u}_n &= [a_{1n} \ a_{2n} \ \cdots \ a_{mn}]^T = b_{1n}\mathbf{c}_1 + b_{2n}\mathbf{c}_2 + \cdots + b_{rn}\mathbf{c}_r.
 \end{aligned}$$

Each column vector of A is a linear combination of r vectors, so you know that the dimension of the column space of A is less than or equal to r (the dimension of the row space of A). That is,

$$\dim(\text{column space of } A) \leq \dim(\text{row space of } A).$$

Repeating this procedure for A^T , you can conclude that the dimension of the column space of A^T is less than or equal to the dimension of the row space of A^T . But this implies that the dimension of the row space of A is less than or equal to the dimension of the column space of A . That is,

$$\dim(\text{row space of } A) \leq \dim(\text{column space of } A).$$

REMARK

Some texts distinguish between the *row rank* and the *column rank* of a matrix, but these ranks are equal (Theorem 4.15). So, this text will not distinguish between them.

So, the two dimensions must be equal.

The dimension of the row (or column) space of a matrix is the **rank** of the matrix.

Definition of the Rank of a Matrix

The dimension of the row (or column) space of a matrix A is the **rank** of A and is denoted by $\text{rank}(A)$.

EXAMPLE 6

Finding the Rank of a Matrix

To find the rank of the matrix A below, convert to a matrix B in row-echelon form as shown.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix B has three nonzero rows, so the rank of A is 3.

THE NULLSPACE OF A MATRIX

Row and column spaces and rank have some important applications to systems of linear equations. Consider first the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix, $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$ is the column vector of variables, and $\mathbf{0} = [0 \ 0 \ \dots \ 0]^T$ is the zero vector in R^m . The next theorem tells you that the set of all solutions of this homogeneous system is a subspace of R^n .

REMARK
 The nullspace of A is also called the **solution space** of the system $A\mathbf{x} = \mathbf{0}$.

THEOREM 4.16 Solutions of a Homogeneous System
 If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the **nullspace** of A and is denoted by $N(A)$. So,

$$N(A) = \{\mathbf{x} \in R^n: A\mathbf{x} = \mathbf{0}\}.$$
 The dimension of the nullspace of A is the **nullity** of A .

PROOF

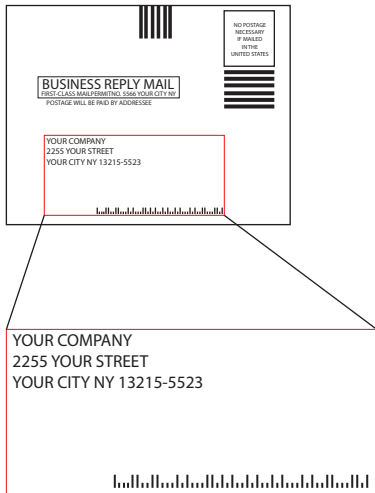
The size of A is $m \times n$, so you know that $\mathbf{x}$ has size $n \times 1$, and the set of all solutions of the system is a *subset* of R^n . This set is clearly nonempty, because $A\mathbf{0} = \mathbf{0}$. Verify that it is a subspace by showing that it is closed under the operations of addition and scalar multiplication. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solution vectors of the system $A\mathbf{x} = \mathbf{0}$, and let c be a scalar. Both $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$, so you know that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
Addition

and

$$A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\mathbf{0}) = \mathbf{0}.$$
Scalar multiplication

So, both $(\mathbf{x}_1 + \mathbf{x}_2)$ and $c\mathbf{x}_1$ are solutions of $A\mathbf{x} = \mathbf{0}$, and you can conclude that the set of all solutions forms a subspace of R^n .



LINEAR ALGEBRA APPLIED
 The U.S. Postal Service uses barcodes to represent such information as ZIP codes and delivery addresses. The ZIP + 4 barcode shown at the left starts with a long bar, then has a sequence of short and long bars to represent each digit in the ZIP + 4 code, an additional digit for error checking, and then the code ends with a long bar. The code for the digits is shown below.

0 =	1 =	2 =	3 =	4 =
5 =	6 =	7 =	8 =	9 =

The error checking digit is such that when it is summed with the digits in the ZIP + 4 code, the result is a multiple of 10. (Verify this, as well as whether the ZIP + 4 code shown is coded correctly.) More sophisticated barcodes will also include error correcting digit(s). In an analogous way, matrices can be used to check for errors in transmitted messages. Information in the form of column vectors can be multiplied by an error detection matrix. When the resulting product is in the nullspace of the error detection matrix, no error in transmission exists. Otherwise, an error exists somewhere in the message. If the error detection matrix also has error correction, then the resulting matrix product will also tell where the error is occurring.

EXAMPLE 7**Finding the Nullspace of a Matrix**

Find the nullspace of the matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

SOLUTION

The nullspace of A is the solution space of the homogeneous system

$$A\mathbf{x} = \mathbf{0}.$$

To solve this system, you could write the augmented matrix $[A \ \mathbf{0}]$ in reduced row-echelon form. However, the last column of the augmented matrix consists entirely of zeros and will not change as you perform row operations, so it is sufficient to find the reduced row-echelon form of A .

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations corresponding to the reduced row-echelon form is

$$\begin{aligned} x_1 + 2x_2 + 3x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

Choose x_2 and x_4 as free variables to represent the solutions in parametric form.

$$x_1 = -2s - 3t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t$$

This means that the solution space of $A\mathbf{x} = \mathbf{0}$ consists of all solution vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

So, a basis for the nullspace of A consists of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

In other words, these two vectors are solutions of $A\mathbf{x} = \mathbf{0}$, and all linear combinations of these two vectors are also solutions.

In Example 7, matrix A has four columns. Furthermore, the rank of A is 2, and the dimension of the nullspace is 2. So,

$$\text{Number of columns} = \text{rank} + \text{nullity}.$$

One way to see this is to look at the reduced row-echelon form of A .

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns with the leading 1's (columns 1 and 3) determine the rank of the matrix. The other columns (2 and 4) determine the nullity of the matrix because they correspond to the free variables. The next theorem generalizes this relationship.

REMARK

Although Example 7 shows that the basis spans the solution set, it does not show that the vectors in the basis are linearly independent. When you solve homogeneous systems from the reduced row-echelon form, the spanning set is always linearly independent. Verify this for the basis found in Example 7.

THEOREM 4.17 Dimension of the Solution Space

If A is an $m \times n$ matrix of rank r , then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is, $n = \text{rank}(A) + \text{nullity}(A)$.

PROOF

A has rank r , so you know it is row-equivalent to a reduced row-echelon matrix B with r nonzero rows. No generality is lost by assuming that the upper left corner of B has the form of the $r \times r$ identity matrix I_r . Moreover, the zero rows of B contribute nothing to the solution, so discard them to form the $r \times n$ matrix B' , where B' is the augmented matrix $[I_r \ C]$. The matrix C has $n - r$ columns corresponding to the variables $x_{r+1}, x_{r+2}, \dots, x_n$, and the solution space of $A\mathbf{x} = \mathbf{0}$ can be represented by the system

$$\begin{array}{rcl} x_1 + & c_{11}x_{r+1} + c_{12}x_{r+2} + \cdots + c_{1,n-r}x_n & = 0 \\ x_2 + & c_{21}x_{r+1} + c_{22}x_{r+2} + \cdots + c_{2,n-r}x_n & = 0 \\ & \vdots & \\ x_r + & c_{r1}x_{r+1} + c_{r2}x_{r+2} + \cdots + c_{r,n-r}x_n & = 0. \end{array}$$

Solving for the first r variables in terms of the last $n - r$ variables produces $n - r$ vectors in the basis for the solution space, so the solution space has dimension $n - r$. 

Example 8 illustrates this theorem and further explores the column space of a matrix.

EXAMPLE 8**Rank, Nullity of a Matrix, and Basis for the Column Space**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, and $\mathbf{a}_5$. Find (a) the rank and nullity of A , and (b) a subset of the column vectors of A that forms a basis for the column space of A .

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5$

SOLUTION

Let B be the reduced row-echelon form of A .

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- a.** B has three nonzero rows, so the rank of A is 3. Also, the number of columns of A is $n = 5$, which implies that the nullity of A is $n - \text{rank} = 5 - 3 = 2$.
- b.** The first, second, and fourth column vectors of B are linearly independent, so the corresponding column vectors of A ,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the column space of A . 

SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

You now know that the set of all solution vectors of the *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ is a subspace. The set of all solution vectors of the *nonhomogeneous* system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$, is *not* a subspace because the zero vector is never a solution of a nonhomogeneous system. There is a relationship, however, between the sets of solutions of the two systems $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Specifically, if $\mathbf{x}_p$ is a *particular* solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then *every* solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. The next theorem states this important concept.

THEOREM 4.18 Solutions of a Nonhomogeneous Linear System

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

PROOF

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. Then $(\mathbf{x} - \mathbf{x}_p)$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, because

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Letting $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, you have $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$. 

EXAMPLE 9

Finding the Solution Set of a Nonhomogeneous System

Find the set of all solution vectors of the system of linear equations.

$$\begin{array}{rrcr} x_1 & & -2x_3 & + x_4 = 5 \\ 3x_1 & + x_2 & -5x_3 & = 8 \\ x_1 & + 2x_2 & & -5x_4 = -9 \end{array}$$

SOLUTION

The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ reduces as shown below.

$$\left[\begin{array}{ccccc} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array} \right] \rightarrow \left[\begin{array}{ccccc} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The system of linear equations corresponding to the reduced row-echelon matrix is

$$\begin{array}{rrcr} x_1 & & -2x_3 & + x_4 = 5 \\ & x_2 & + x_3 & - 3x_4 = -7. \end{array}$$

Letting $x_3 = s$ and $x_4 = t$, write a representative solution vector of $A\mathbf{x} = \mathbf{b}$ as shown below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 + 2s - t \\ -7 - s + 3t \\ 0 + s + 0t \\ 0 + 0s + t \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \mathbf{x}_p + s\mathbf{u}_1 + t\mathbf{u}_2$$

$\mathbf{x}_p$ is a *particular* solution vector of $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ represents an arbitrary vector in the solution space of $A\mathbf{x} = \mathbf{0}$. 

The next theorem describes how the column space of a matrix can be used to determine whether a system of linear equations is consistent.

THEOREM 4.19 Solutions of a System of Linear Equations

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A .

PROOF

For the system $A\mathbf{x} = \mathbf{b}$, let A , $\mathbf{x}$, and $\mathbf{b}$ be the $m \times n$ coefficient matrix, the $n \times 1$ column matrix of variables, and the $m \times 1$ right-hand side, respectively. Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

So, $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b} = [b_1 \ b_2 \ \cdots \ b_m]^T$ is a linear combination of the columns of A . That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of R^m spanned by the columns of A . 

EXAMPLE 10

Consistency of a System of Linear Equations

REMARK

The reduced row-echelon form of $[A \ \mathbf{b}]$ is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(verify this). So, there are infinitely many ways to write $\mathbf{b}$ as a linear combination of the columns of A .

Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1. \end{aligned}$$

The augmented matrix for the system is

$$[A \ \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$

Notice that $\mathbf{b} = 2\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_3$. So, $\mathbf{b}$ is in the column space of A , and the system of linear equations is consistent. 

The summary below presents several major results involving systems of linear equations, matrices, determinants, and vector spaces.

Summary of Equivalent Conditions for Square Matrices

If A is an $n \times n$ matrix, then the conditions below are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n .
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The n row vectors of A are linearly independent.
8. The n column vectors of A are linearly independent.

4.6 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Row Vectors and Column Vectors In Exercises 1–4, write (a) the row vectors and (b) the column vectors of the matrix.

1. $\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 5 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 3 & 1 \\ 1 & -4 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 3 & -4 \\ 4 & 0 & -1 \\ -6 & 1 & 1 \end{bmatrix}$

Finding a Basis for a Row Space and Rank In Exercises 5–12, find (a) a basis for the row space and (b) the rank of the matrix.

5. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 5 \\ -2 & -5 \\ -6 & -15 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 6 & 18 \\ 7 & 40 & 116 \\ -3 & -12 & -27 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 0 & 2 & 3 & 1 \\ 2 & -1 & 2 & 0 & 1 \\ 5 & 2 & 2 & 1 & -1 \\ 4 & 0 & 2 & 2 & 1 \\ 2 & -2 & 0 & 0 & 1 \end{bmatrix}$

Finding a Basis for a Subspace In Exercises 13–16, find a basis for the subspace of $\mathbb{R}^3$ spanned by S .

13. $S = \{(1, 2, 4), (-1, 3, 4), (2, 3, 1)\}$

14. $S = \{(2, 3, -1), (1, 3, -9), (0, 1, 5)\}$

15. $S = \{(4, 4, 8), (1, 1, 2), (1, 1, 1)\}$

16. $S = \{(1, 2, 2), (-1, 0, 0), (1, 1, 1)\}$

Finding a Basis for a Subspace In Exercises 17–20, find a basis for the subspace of $\mathbb{R}^4$ spanned by S .

17. $S = \{(2, 9, -2, 53), (-3, 2, 3, -2), (8, -3, -8, 17), (0, -3, 0, 15)\}$

18. $S = \{(6, -3, 6, 34), (3, -2, 3, 19), (8, 3, -9, 6), (-2, 0, 6, -5)\}$

19. $S = \{(-3, 2, 5, 28), (-6, 1, -8, -1), (14, -10, 12, -10), (0, 5, 12, 50)\}$

20. $S = \{(2, 5, -3, -2), (-2, -3, 2, -5), (1, 3, -2, 2), (-1, -5, 3, 5)\}$

Finding a Basis for a Column Space and Rank In Exercises 21–26, find (a) a basis for the column space and (b) the rank of the matrix.

21. $\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix}$

25. $\begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 4 & -2 & 1 & 1 \\ 2 & 5 & 4 & -2 & 2 \\ 4 & 3 & 1 & 1 & 2 \\ 2 & -4 & 2 & -1 & 1 \\ 0 & 1 & 4 & 2 & -1 \end{bmatrix}$

Finding the Nullspace of a Matrix In Exercises 27–40, find the nullspace of the matrix.

27. $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

28. $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

29. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

33. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{bmatrix}$

34. $A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$

35. $A = \begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 2 & 1 \end{bmatrix}$

36. $A = \begin{bmatrix} -16 & 1 \\ 48 & -3 \\ -80 & 5 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}$

39. $A = \begin{bmatrix} 2 & 6 & 3 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & -2 & 1 & 1 \\ 0 & 6 & 2 & 0 \end{bmatrix}$

40. $A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}$

Rank, Nullity, Bases, and Linear Independence In Exercises 41 and 42, use the fact that matrices A and B are row-equivalent.

- Find the rank and nullity of A .
- Find a basis for the nullspace of A .
- Find a basis for the row space of A .
- Find a basis for the column space of A .
- Determine whether the rows of A are linearly independent.
- Let the columns of A be denoted by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, and $\mathbf{a}_5$. Determine whether each set is linearly independent.
 - $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$
 - $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$
 - $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$

$$41. A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$42. A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding a Basis and Dimension In Exercises 43–48, find (a) a basis for and (b) the dimension of the solution space of the homogeneous system of linear equations.

- $-x + y + z = 0$
 $3x - y = 0$
 $2x - 4y - 5z = 0$
- $x - 2y + 3z = 0$
 $-3x + 6y - 9z = 0$
- $3x_1 + 3x_2 + 15x_3 + 11x_4 = 0$
 $x_1 - 3x_2 + x_3 + x_4 = 0$
 $2x_1 + 3x_2 + 11x_3 + 8x_4 = 0$
- $2x_1 + 2x_2 + 4x_3 - 2x_4 = 0$
 $x_1 + 2x_2 + x_3 + 2x_4 = 0$
 $-x_1 + x_2 + 4x_3 - 2x_4 = 0$
- $9x_1 - 4x_2 - 2x_3 - 20x_4 = 0$
 $12x_1 - 6x_2 - 4x_3 - 29x_4 = 0$
 $3x_1 - 2x_2 - 7x_4 = 0$
 $3x_1 - 2x_2 - x_3 - 8x_4 = 0$
- $x_1 + 3x_2 + 2x_3 + 22x_4 + 13x_5 = 0$
 $x_1 + x_3 - 2x_4 + x_5 = 0$
 $3x_1 + 6x_2 + 5x_3 + 42x_4 + 27x_5 = 0$

Nonhomogeneous System In Exercises 49–56, determine whether the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent. If it is, write the solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$.

- $x - 4y = 17$
 $3x - 12y = 51$
 $-2x + 8y = -34$
- $x + 2y - 4z = -1$
 $-3x - 6y + 12z = 3$
- $x + 3y + 10z = 18$
 $-2x + 7y + 32z = 29$
 $-x + 3y + 14z = 12$
 $x + y + 2z = 8$
- $2x - 4y + 5z = 8$
 $-7x + 14y + 4z = -28$
 $3x - 6y + z = 12$
- $3x - 8y + 4z = 19$
 $-6y + 2z + 4w = 5$
 $5x + 22z + w = 29$
 $x - 2y + 2z = 8$
- $3w - 2x + 16y - 2z = -7$
 $-w + 5x - 14y + 18z = 29$
 $3w - x + 14y + 2z = 1$
- $x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 0$
 $-5x_1 - 10x_2 + 3x_3 + 3x_4 + 55x_5 = -8$
 $x_1 + 2x_2 + 2x_3 - 3x_4 - 5x_5 = 14$
 $-x_1 - 2x_2 + x_3 + x_4 + 15x_5 = -2$
- $5x_1 - 4x_2 + 12x_3 - 33x_4 + 14x_5 = -4$
 $-2x_1 + x_2 - 6x_3 + 12x_4 - 8x_5 = 1$
 $2x_1 - x_2 + 6x_3 - 12x_4 + 8x_5 = -1$

Consistency of $A\mathbf{x} = \mathbf{b}$ In Exercises 57–62, determine whether $\mathbf{b}$ is in the column space of A . If it is, write $\mathbf{b}$ as a linear combination of the column vectors of A .

- $A = \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
- $A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$
- $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$
- $A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & -2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -9 \\ 11 \\ -25 \end{bmatrix}$

- 63. Proof** Prove that if A is not square, then either the row vectors of A or the column vectors of A form a linearly dependent set.
- 64.** Give an example showing that the rank of the product of two matrices can be less than the rank of either matrix.
- 65.** Give examples of matrices A and B of the same size such that
- $\text{rank}(A + B) < \text{rank}(A)$ and $\text{rank}(A + B) < \text{rank}(B)$
 - $\text{rank}(A + B) = \text{rank}(A)$ and $\text{rank}(A + B) = \text{rank}(B)$
 - $\text{rank}(A + B) > \text{rank}(A)$ and $\text{rank}(A + B) > \text{rank}(B)$.
- 66. Proof** Prove that the nonzero row vectors of a matrix in row-echelon form are linearly independent.
- 67.** Let A be an $m \times n$ matrix (where $m < n$) whose rank is r .
- What is the largest value r can be?
 - How many vectors are in a basis for the row space of A ?
 - How many vectors are in a basis for the column space of A ?
 - Which vector space R^k has the row space as a subspace?
 - Which vector space R^k has the column space as a subspace?
- 68.** Show that the three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) in a plane are collinear if and only if the matrix
- $$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$
- has rank less than 3.
- 69.** Consider an $m \times n$ matrix A and an $n \times p$ matrix B . Show that the row vectors of AB are in the row space of B and the column vectors of AB are in the column space of A .
- 70.** Find the rank of the matrix
- $$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & n+3 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \dots & n^2 \end{bmatrix}$$
- for $n = 2, 3$, and 4 . Can you find a pattern in these ranks?
- 71. Proof** Prove each property of the system of linear equations in n variables $A\mathbf{x} = \mathbf{b}$.
- If $\text{rank}(A) = \text{rank}([A \ \mathbf{b}]) = n$, then the system has a unique solution.
 - If $\text{rank}(A) = \text{rank}([A \ \mathbf{b}]) < n$, then the system has infinitely many solutions.
 - If $\text{rank}(A) < \text{rank}([A \ \mathbf{b}])$, then the system is inconsistent.
- 72. Proof** Let A be an $m \times n$ matrix. Prove that $N(A) \subset N(A^T A)$.

True or False? In Exercises 73–76, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 73.** (a) The nullspace of a matrix A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
 (b) The dimension of the nullspace of a matrix A is the nullity of A .
- 74.** (a) If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B , then the row space of A is equivalent to the row space of B .
 (b) If A is an $m \times n$ matrix of rank r , then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $m - r$.
- 75.** (a) If an $m \times n$ matrix B can be obtained from elementary row operations on an $m \times n$ matrix A , then the column space of B is equal to the column space of A .
 (b) The system of linear equations $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is in the column space of A .
- 76.** (a) The column space of a matrix A is equal to the row space of A^T .
 (b) The row space of a matrix A is equal to the column space of A^T .
- 77.** Let A and B be square matrices of order n satisfying $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ in R^n .
- Find the rank and nullity of $A - B$.
 - Show that A and B must be identical.

78. CAPSTONE The dimension of the row space of a 3×5 matrix A is 2.

- What is the dimension of the column space of A ?
- What is the rank of A ?
- What is the nullity of A ?
- What is the dimension of the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$?

- 79. Proof** Let A be an $m \times n$ matrix.
- Prove that the system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent for all column vectors $\mathbf{b}$ if and only if the rank of A is m .
 - Prove that the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the columns of A are linearly independent.
- 80. Proof** Prove that row operations do not change the dependency relationships among the columns of an $m \times n$ matrix.
- 81. Writing** Explain why the row vectors of a 4×3 matrix form a linearly dependent set. (Assume all matrix entries are distinct.)

4.7 Coordinates and Change of Basis

-  Find a coordinate matrix relative to a basis in R^n .
-  Find the transition matrix from the basis B to the basis B' in R^n .
-  Represent coordinates in general n -dimensional spaces.

COORDINATE REPRESENTATION IN R^n

In Theorem 4.9, you saw that if B is a basis for a vector space V , then every vector $\mathbf{x}$ in V can be expressed in one and only one way as a linear combination of vectors in B . The coefficients in the linear combination are the **coordinates of $\mathbf{x}$ relative to B** . In the context of coordinates, the order of the vectors in the basis is important, so this will sometimes be emphasized by referring to the basis B as an *ordered* basis.

Coordinate Representation Relative to a Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \dots, c_n$ are the **coordinates of $\mathbf{x}$ relative to the basis B** . The **coordinate matrix** (or **coordinate vector**) of $\mathbf{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

In R^n , column notation is used for the coordinate matrix. For the vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the x_i 's are the coordinates of $\mathbf{x}$ relative to the standard basis S for R^n . So, you have

$$[\mathbf{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

EXAMPLE 1

Coordinates and Components in R^n

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3 relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

SOLUTION

The vector $\mathbf{x}$ can be written as $\mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$, so the coordinate matrix of $\mathbf{x}$ relative to the standard basis is simply

$$[\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

The components of $\mathbf{x}$ are the same as its coordinates relative to the standard basis. 

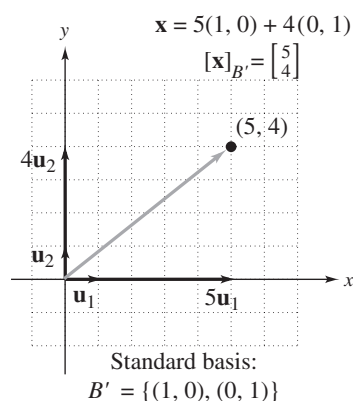
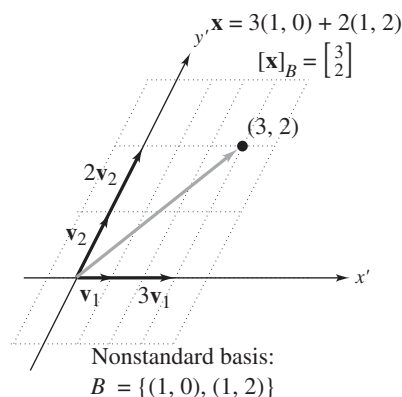


Figure 4.17

EXAMPLE 2**Finding a Coordinate Matrix Relative to a Standard Basis**

The coordinate matrix of $\mathbf{x}$ in R^2 relative to the (nonstandard) ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 0), (1, 2)\}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the coordinate matrix of $\mathbf{x}$ relative to the standard basis $B' = \{\mathbf{u}_1, \mathbf{u}_2\} = \{(1, 0), (0, 1)\}$.

SOLUTION

The coordinate matrix of $\mathbf{x}$ relative to B is $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so

$$\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 = 3(1, 0) + 2(1, 2) = (5, 4) = 5(1, 0) + 4(0, 1).$$

It follows that the coordinate matrix of $\mathbf{x}$ relative to B' is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Figure 4.17 compares these two coordinate representations.

Example 2 shows that the procedure for finding the coordinate matrix relative to a *standard* basis is straightforward. It is more difficult, however, to find the coordinate matrix relative to a *nonstandard* basis. Here is an example.

EXAMPLE 3**Finding a Coordinate Matrix Relative to a Nonstandard Basis**

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3 relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}.$$

SOLUTION

Begin by writing $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

Equating corresponding components produces the system of linear equations and corresponding matrix equation below.

$$c_1 + 2c_3 = 1$$

$$-c_2 + 3c_3 = 2$$

$$c_1 + 2c_2 - 5c_3 = -1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The solution of this system is $c_1 = 5$, $c_2 = -8$, and $c_3 = -2$. So,

$$\mathbf{x} = 5(1, 0, 1) + (-8)(0, -1, 2) + (-2)(2, 3, -5)$$

and the coordinate matrix of $\mathbf{x}$ relative to B' is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

REMARK

It would be incorrect to write the coordinate matrix as

$$\mathbf{x} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

Do you see why?

CHANGE OF BASIS IN R^n

The procedure demonstrated in Examples 2 and 3 is called a **change of basis**. That is, you were given the coordinates of a vector relative to a basis B and were asked to find the coordinates relative to another basis B' .

For instance, if in Example 3 you let B be the standard basis, then the problem of finding the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ relative to the basis B' becomes one of solving for c_1, c_2 , and c_3 in the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}}_P \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\mathbf{x}]_{B'}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\mathbf{x}]_B}.$$

The matrix P is the *transition matrix from B' to B* , where $[\mathbf{x}]_{B'}$ is the coordinate matrix of $\mathbf{x}$ relative to B' , and $[\mathbf{x}]_B$ is the coordinate matrix of $\mathbf{x}$ relative to B . Multiplication by the transition matrix P changes a coordinate matrix relative to B' into a coordinate matrix relative to B . That is,

$$P[\mathbf{x}]_{B'} = [\mathbf{x}]_B. \quad \text{Change of basis from } B' \text{ to } B$$

To perform a change of basis from B to B' , use the matrix P^{-1} (the *transition matrix from B to B'*) and write

$$[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B. \quad \text{Change of basis from } B \text{ to } B'$$

So, the change of basis problem in Example 3 can be represented by the matrix equation

$$\underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\mathbf{x}]_{B'}} = \underbrace{\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\mathbf{x}]_B} = \underbrace{\begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}}_{[\mathbf{x}]_{B'}}.$$

Generalizing this discussion, assume that

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \text{and} \quad B' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

are two ordered bases for R^n . If $\mathbf{x}$ is a vector in R^n and

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{B'} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

are the coordinate matrices of $\mathbf{x}$ relative to B and B' , then the **transition matrix P from B' to B** is the matrix P such that

$$[\mathbf{x}]_B = P[\mathbf{x}]_{B'}.$$

The next theorem tells you that the transition matrix P is invertible and its inverse is the **transition matrix from B to B'** . That is,

$$[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B.$$

THEOREM 4.20 The Inverse of a Transition Matrix

If P is the transition matrix from a basis B' to a basis B in R^n , then P is invertible and the transition matrix from B to B' is P^{-1} .

Before proving Theorem 4.20, it is necessary to look at and prove a preliminary lemma.

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be two bases for a vector space V . If

$$\begin{aligned}\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \dots + c_{n1}\mathbf{u}_n \\ \mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \dots + c_{n2}\mathbf{u}_n \\ &\vdots \\ \mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \dots + c_{nn}\mathbf{u}_n\end{aligned}$$

then the transition matrix from B to B' is

$$Q = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}.$$

PROOF (OF LEMMA)

Let $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n$ be an arbitrary vector in V . The coordinate matrix of $\mathbf{v}$ with respect to the basis B is

$$[\mathbf{v}]_B = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

Then you have

$$Q[\mathbf{v}]_B = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \dots + c_{1n}d_n \\ c_{21}d_1 + c_{22}d_2 + \dots + c_{2n}d_n \\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \dots + c_{nn}d_n \end{bmatrix}.$$

On the other hand,

$$\begin{aligned}\mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n \\ &= d_1(c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \dots + c_{n1}\mathbf{u}_n) + d_2(c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \dots + c_{n2}\mathbf{u}_n) + \dots \\ &\quad + d_n(c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \dots + c_{nn}\mathbf{u}_n) \\ &= (d_1c_{11} + d_2c_{12} + \dots + d_nc_{1n})\mathbf{u}_1 + (d_1c_{21} + d_2c_{22} + \dots + d_nc_{2n})\mathbf{u}_2 + \dots \\ &\quad + (d_1c_{n1} + d_2c_{n2} + \dots + d_nc_{nn})\mathbf{u}_n\end{aligned}$$

which implies

$$[\mathbf{v}]_{B'} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \dots + c_{1n}d_n \\ c_{21}d_1 + c_{22}d_2 + \dots + c_{2n}d_n \\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \dots + c_{nn}d_n \end{bmatrix}.$$

So, $Q[\mathbf{v}]_B = [\mathbf{v}]_{B'}$ and you can conclude that Q is the transition matrix from B to B' . 

PROOF (OF THEOREM 4.20)

From the preceding lemma, let Q be the transition matrix from B to B' . Then $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_{B'} = Q[\mathbf{v}]_B$, which implies that $[\mathbf{v}]_B = PQ[\mathbf{v}]_B$ for every vector $\mathbf{v}$ in R^n . From this it follows that $PQ = I$. So, P is invertible and P^{-1} is equal to Q , the transition matrix from B to B' . 

REMARK

Verify that the transition matrix P^{-1} from B to B' is $(B')^{-1}B$. Also verify that the transition matrix P from B' to B is $B^{-1}B'$. You can use these relationships to check the results obtained by Gauss-Jordan elimination.

Gauss-Jordan elimination can be used to find the transition matrix P^{-1} . First define two matrices B and B' whose columns correspond to the vectors in B and B' . That is,

$$B = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix}.$$

$\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n$
 $\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n$

Then, by reducing the $n \times 2n$ matrix $[B' \ B]$ so that the identity matrix I_n occurs in place of B' , you obtain the matrix $[I_n \ P^{-1}]$. The next theorem states this procedure formally.

THEOREM 4.21 Transition Matrix from B to B'

Let

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \text{and} \quad B' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

be two bases for R^n . Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \ B]$, as shown below.

$$[B' \ B] \xrightarrow{\text{Gauss-Jordan}} [I_n \ P^{-1}]$$

PROOF

To begin, let

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n \\ \mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n \\ &\vdots \\ \mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n \end{aligned}$$

which implies that

$$c_{1i} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} + c_{2i} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{n2} \end{bmatrix} + \cdots + c_{ni} \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix} = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix}$$

for $i = 1, 2, \dots, n$. From these vector equations, write the n systems of linear equations

$$\begin{aligned} u_{11}c_{1i} + u_{12}c_{2i} + \cdots + u_{1n}c_{ni} &= v_{1i} \\ u_{21}c_{1i} + u_{22}c_{2i} + \cdots + u_{2n}c_{ni} &= v_{2i} \\ &\vdots \\ u_{n1}c_{1i} + u_{n2}c_{2i} + \cdots + u_{nn}c_{ni} &= v_{ni} \end{aligned}$$

for $i = 1, 2, \dots, n$. Each of the n systems has the same coefficient matrix, so you can reduce all n systems simultaneously using the augmented matrix below.

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} & v_{11} & v_{12} & \cdots & v_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} & v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} & v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$$

B'
 B

Applying Gauss-Jordan elimination to this matrix produces

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 1 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

By the lemma following Theorem 4.20, however, the right-hand side of this matrix is $Q = P^{-1}$, which implies that the matrix has the form $[I \ P^{-1}]$, which proves the theorem. 

In the next example, you will apply this procedure to the change of basis problem from Example 3.

EXAMPLE 4

Finding a Transition Matrix

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the transition matrix from B to B' for the bases for R^3 below.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

SOLUTION

First use the vectors in the two bases to form the matrices B and B' .

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Then form the matrix $[B' \ B]$ and use Gauss-Jordan elimination to rewrite $[B' \ B]$ as $[I_3 \ P^{-1}]$.

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{bmatrix}$$

From this, you can conclude that the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

Multiply P^{-1} by the coordinate matrix of $\mathbf{x} = [1 \ 2 \ -1]^T$ to see that the result is the same as that obtained in Example 3. 

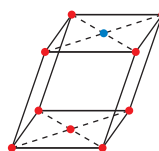
DISCOVERY

- Let $B = \{(1, 0), (1, 2)\}$ and $B' = \{(1, 0), (0, 1)\}$. Form the matrix $[B' \ B]$.
- Make a conjecture about the necessity of using Gauss-Jordan elimination to obtain the transition matrix P^{-1} when the change of basis is from a nonstandard basis to a standard basis.



LINEAR ALGEBRA APPLIED

Crystallography is the science of atomic and molecular structure. In a crystal, atoms are in a repeating pattern called a *lattice*. The simplest repeating unit in a lattice is a *unit cell*. Crystallographers can use bases and coordinate matrices in R^3 to designate the locations of atoms in a unit cell. For example, the figure below shows the unit cell known as *end-centered monoclinic*.



One possible coordinate matrix for the top end-centered (blue) atom is $[\mathbf{x}]_{B'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T$.

Brazhnykov Andriy/Shutterstock.com

Note that when B is the standard basis, as in Example 4, the process of changing $[B' \ B]$ to $[I_n \ P^{-1}]$ becomes

$$[B' \ I_n] \rightarrow [I_n \ P^{-1}].$$

But this is the same process that was used to find inverse matrices in Section 2.3. In other words, if B is the standard basis for R^n , then the transition matrix from B to B' is

$$P^{-1} = (B')^{-1}. \quad \text{Standard basis to nonstandard basis}$$

The process is even simpler when B' is the standard basis, because the matrix $[B' \ B]$ is already in the form

$$[I_n \ B] = [I_n \ P^{-1}].$$

In this case, the transition matrix is simply

$$P^{-1} = B. \quad \text{Nonstandard basis to standard basis}$$

For instance, the transition matrix in Example 2 from $B = \{(1, 0), (1, 2)\}$ to $B' = \{(1, 0), (0, 1)\}$ is

$$P^{-1} = B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

TECHNOLOGY

Many graphing utilities and software programs can form an augmented matrix and find its reduced row-echelon form. If you use a graphing utility, then you may see something similar to the screen below for Example 5.

```

B          [[-3  4 ]
BPRIME     [ 2 -2]]
           [[-1  2 ]
aug(BPRIME,B) [ 2 -2]]
           [[-1  2 -3  4 ]
           [ 2 -2  2 -2]]
rref aug(BPRIME,B)
           [[1  0 -1  2]
           [ 0  1 -2  3]]
  
```

The **Technology Guide** at *CengageBrain.com* can help you use technology to find a transition matrix.

EXAMPLE 5 Finding a Transition Matrix

Find the transition matrix from B to B' for the bases for R^2 below.

$$B = \{(-3, 2), (4, -2)\} \quad \text{and} \quad B' = \{(-1, 2), (2, -2)\}$$

SOLUTION

Begin by forming the matrix

$$[B' \ B] = \begin{bmatrix} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the transition matrix P^{-1} from B to B' :

$$[I_2 \ P^{-1}] = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{bmatrix}.$$

So, you have

$$P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}.$$

In Example 5, if you had found the transition matrix from B' to B (rather than from B to B'), then you would have obtained

$$[B \ B'] = \begin{bmatrix} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{bmatrix}$$

which reduces to

$$[I_2 \ P] = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

The transition matrix from B' to B is

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}.$$

Verify that this is the inverse of the transition matrix found in Example 5 by multiplying PP^{-1} to obtain I_2 .

COORDINATE REPRESENTATION IN GENERAL n -DIMENSIONAL SPACES

One benefit of coordinate representation is that it enables you to represent vectors in any n -dimensional space using the same notation used in R^n . For instance, in Example 6, note that the coordinate matrix of a vector in P_3 is a vector in R^4 .

EXAMPLE 6

Coordinate Representation in P_3

Find the coordinate matrix of

$$p = 4 - 2x^2 + 3x^3$$

relative to the standard basis for P_3 ,

$$S = \{1, x, x^2, x^3\}.$$

SOLUTION

Write p as a linear combination of the basis vectors (in the given order).

$$p = 4(1) + 0(x) + (-2)(x^2) + 3(x^3)$$

So, the coordinate matrix of p relative to S is

$$[p]_S = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}.$$

In the next example, the coordinate matrix of a vector in $M_{3,1}$ is a vector in R^3 .

EXAMPLE 7

Coordinate Representation in $M_{3,1}$

Find the coordinate matrix of

$$X = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

relative to the standard basis for $M_{3,1}$,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

SOLUTION

X can be written as

$$X = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so the coordinate matrix of X relative to S is

$$[X]_S = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

REMARK

In Section 6.2 you will learn more about the use of R^n to represent an arbitrary n -dimensional vector space.

Theorems 4.20 and 4.21 can be generalized to cover arbitrary n -dimensional spaces. This text, however, does not cover the generalizations of these theorems.

4.7 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding a Coordinate Matrix In Exercises 1–4, find the coordinate matrix of $\mathbf{x}$ in R^n relative to the standard basis.

1. $\mathbf{x} = (5, -2)$ 2. $\mathbf{x} = (1, -3, 0)$
 3. $\mathbf{x} = (7, -4, -1, 2)$ 4. $\mathbf{x} = (-6, 12, -4, 9, -8)$

Finding a Coordinate Matrix In Exercises 5–10, given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis B for R^n , find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

5. $B = \{(2, -1), (0, 1)\}$, 6. $B = \{(-2, 3), (3, -2)\}$,

$$[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

7. $B = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$,

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

8. $B = \left\{ \left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2} \right), \left(3, 4, \frac{7}{2} \right), \left(-\frac{3}{2}, 6, 2 \right) \right\}$,

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

9. $B = \{(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$,

$$[\mathbf{x}]_B = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

10. $B = \{(4, 0, 7, 3), (0, 5, -1, -1), (-3, 4, 2, 1), (0, 1, 5, 0)\}$,

$$[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

Finding a Coordinate Matrix In Exercises 11–16, find the coordinate matrix of $\mathbf{x}$ in R^n relative to the basis B' .

11. $B' = \{(4, 0), (0, 3)\}$, $\mathbf{x} = (12, 6)$
 12. $B' = \{(-5, 6), (3, -2)\}$, $\mathbf{x} = (-17, 22)$
 13. $B' = \{(8, 11, 0), (7, 0, 10), (1, 4, 6)\}$, $\mathbf{x} = (3, 19, 2)$
 14. $B' = \left\{ \left(\frac{3}{2}, 4, 1 \right), \left(\frac{3}{4}, \frac{5}{2}, 0 \right), \left(1, \frac{1}{2}, 2 \right) \right\}$, $\mathbf{x} = \left(3, -\frac{1}{2}, 8 \right)$
 15. $B' = \{(4, 3, 3), (-11, 0, 11), (0, 9, 2)\}$,
 $\mathbf{x} = (11, 18, -7)$
 16. $B' = \{(9, -3, 15, 4), (3, 0, 0, 1), (0, -5, 6, 8), (3, -4, 2, -3)\}$,
 $\mathbf{x} = (0, -20, 7, 15)$

Finding a Transition Matrix In Exercises 17–24, find the transition matrix from B to B' .

17. $B = \{(1, 0), (0, 1)\}$, $B' = \{(2, 4), (1, 3)\}$
 18. $B = \{(1, 0), (0, 1)\}$, $B' = \{(1, 1), (5, 6)\}$
 19. $B = \{(2, 4), (-1, 3)\}$, $B' = \{(1, 0), (0, 1)\}$
 20. $B = \{(1, 1), (1, 0)\}$, $B' = \{(1, 0), (0, 1)\}$
 21. $B = \{(-1, 0, 0), (0, 1, 0), (0, 0, -1)\}$,
 $B' = \{(0, 0, 2), (1, 4, 0), (5, 0, 2)\}$
 22. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,
 $B' = \{(1, 3, -1), (2, 7, -4), (2, 9, -7)\}$
 23. $B = \{(3, 4, 0), (-2, -1, 1), (1, 0, -3)\}$,
 $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
 24. $B = \{(1, 3, 2), (2, -1, 2), (5, 6, 1)\}$,
 $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$



Finding a Transition Matrix In Exercises 25–36, use a software program or a graphing utility to find the transition matrix from B to B' .

25. $B = \{(2, 5), (1, 2)\}$, $B' = \{(2, 1), (-1, 2)\}$
 26. $B = \{(-2, 1), (3, 2)\}$, $B' = \{(1, 2), (-1, 0)\}$
 27. $B = \{(-3, 4), (3, -5)\}$, $B' = \{(-5, -6), (7, -8)\}$
 28. $B = \{(2, -2), (-2, -2)\}$, $B' = \{(3, -3), (-3, -3)\}$
 29. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,
 $B' = \{(1, 3, 3), (1, 5, 6), (1, 4, 5)\}$
 30. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,
 $B' = \{(2, -1, 4), (0, 2, 1), (-3, 2, 1)\}$
 31. $B = \{(1, 2, 4), (-1, 2, 0), (2, 4, 0)\}$,
 $B' = \{(0, 2, 1), (-2, 1, 0), (1, 1, 1)\}$
 32. $B = \{(3, 2, 1), (1, 1, 2), (1, 2, 0)\}$,
 $B' = \{(1, 1, -1), (0, 1, 2), (-1, 4, 0)\}$
 33. $B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$,
 $B' = \{(1, 3, 2, -1), (-2, -5, -5, 4), (-1, -2, -2, 4), (-2, -3, -5, 11)\}$
 34. $B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$,
 $B' = \{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$
 35. $B = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$,
 $B' = \{(1, 2, 4, -1, 2), (-2, -3, 4, 2, 1), (0, 1, 2, -2, 1), (0, 1, 2, 2, 1), (1, -1, 0, 1, 2)\}$
 36. $B = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$,
 $B' = \{(2, 4, -2, 1, 0), (3, -1, 0, 1, 2), (0, 0, -2, 4, 5), (2, -1, 2, 1, 1), (0, 1, 2, -3, 1)\}$

Finding Transition and Coordinate Matrices In Exercises 37–40, (a) find the transition matrix from B to B' , (b) find the transition matrix from B' to B , (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[\mathbf{x}]_B$, given the coordinate matrix $[\mathbf{x}]_{B'}$.

37. $B = \{(1, 3), (-2, -2)\}$, $B' = \{(-12, 0), (-4, 4)\}$,

$$[\mathbf{x}]_{B'} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

38. $B = \{(2, -2), (6, 3)\}$, $B' = \{(1, 1), (32, 31)\}$,

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

39. $B = \{(1, 0, 2), (0, 1, 3), (1, 1, 1)\}$,
 $B' = \{(2, 1, 1), (1, 0, 0), (0, 2, 1)\}$,

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

40. $B = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$,
 $B' = \{(2, 2, 0), (0, 1, 1), (1, 0, 1)\}$,

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$



Finding Transition and Coordinate Matrices In Exercises 41–44, use a software program or a graphing utility to (a) find the transition matrix from B to B' , (b) find the transition matrix from B' to B , (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[\mathbf{x}]_B$, given the coordinate matrix $[\mathbf{x}]_{B'}$.

41. $B = \{(4, 2, -4), (6, -5, -6), (2, -1, 8)\}$,

$$B' = \{(1, 0, 4), (4, 2, 8), (2, 5, -2)\},$$

$$[\mathbf{x}]_{B'} = [1 \quad -1 \quad 2]^T$$

42. $B = \{(1, 3, 4), (2, -5, 2), (-4, 2, -6)\}$,

$$B' = \{(1, 2, -2), (4, 1, -4), (-2, 5, 8)\},$$

$$[\mathbf{x}]_{B'} = [-1 \quad 0 \quad 2]^T$$

43. $B = \{(2, 0, -1), (0, -1, 3), (1, -3, -2)\}$,

$$B' = \{(0, -1, -3), (-1, 3, -2), (-3, -2, 0)\},$$

$$[\mathbf{x}]_{B'} = [4 \quad -3 \quad -2]^T$$

44. $B = \{(1, -1, 9), (-9, 1, 1), (1, 9, -1)\}$,

$$B' = \{(3, 0, 3), (-3, 3, 0), (0, -3, 3)\},$$

$$[\mathbf{x}]_{B'} = [-5 \quad -4 \quad 1]^T$$

Coordinate Representation in P_3 In Exercises 45–48, find the coordinate matrix of p relative to the standard basis for P_3 .

45. $p = 1 + 5x - 2x^2 + x^3$ 46. $p = -2 - 3x + 4x^3$

47. $p = 13 + 114x + 3x^2$

48. $p = 4 + 11x + x^2 + 2x^3$

Coordinate Representation in $M_{3,1}$ In Exercises 49–52, find the coordinate matrix of X relative to the standard basis for $M_{3,1}$.

49. $X = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

50. $X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

51. $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

52. $X = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$

53. **Writing** Is it possible for a transition matrix to equal the identity matrix? Explain.

54. **CAPSTONE** Let B and B' be two bases for R^n .

- When $B = I_n$, write the transition matrix from B to B' in terms of B' .
- When $B' = I_n$, write the transition matrix from B to B' in terms of B .
- When $B = I_n$, write the transition matrix from B' to B in terms of B' .
- When $B' = I_n$, write the transition matrix from B' to B in terms of B .

True or False? In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- (a) If P is the transition matrix from a basis B to B' , then the equation $P[\mathbf{x}]_{B'} = [\mathbf{x}]_B$ represents the change of basis from B to B' .

(b) If B is the standard basis in R^n , then the transition matrix from B to B' is $P^{-1} = (B')^{-1}$.

(c) For any 4×1 matrix X , the coordinate matrix $[X]_S$ relative to the standard basis for $M_{4,1}$ is equal to X itself.
- (a) If P is the transition matrix from a basis B' to B , then P^{-1} is the transition matrix from B to B' .

(b) To perform the change of basis from a nonstandard basis B' to the standard basis B , the transition matrix P^{-1} is simply B' .

(c) The coordinate matrix of $p = -3 + x + 5x^2$ relative to the standard basis for P_2 is $[p]_S = [5 \quad 1 \quad -3]^T$.
- Let P be the transition matrix from B'' to B' , and let Q be the transition matrix from B' to B . What is the transition matrix from B'' to B ?
- Let P be the transition matrix from B'' to B' , and let Q be the transition matrix from B' to B . What is the transition matrix from B to B'' ?

4.8 Applications of Vector Spaces

-  Use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence.
-  Identify and sketch the graph of a conic section and perform a rotation of axes.

LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A **linear differential equation of order n** is of the form

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = f(x)$$

where $g_0, g_1, \dots, g_{n-1}$ and f are functions of x with a common domain. If $f(x) = 0$, then the equation is **homogeneous**. Otherwise it is **nonhomogeneous**. A function y is a **solution** of the linear differential equation if the equation is satisfied when y and its first n derivatives are substituted into the equation.

EXAMPLE 1

A Second-Order Linear Differential Equation

Show that both $y = e^x$ and $y_2 = e^{-x}$ are solutions of the second-order linear differential equation $y'' - y = 0$.

SOLUTION

For the function $y_1 = e^x$, you have $y_1' = e^x$ and $y_1'' = e^x$. So,

$$y_1'' - y_1 = e^x - e^x = 0$$

which means that $y_1 = e^x$ is a solution of the differential equation. Similarly, for $y_2 = e^{-x}$, you have

$$y_2' = -e^{-x} \quad \text{and} \quad y_2'' = e^{-x}.$$

This implies that

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

So, $y_2 = e^{-x}$ is also a solution of the linear differential equation. 

There are two important observations you can make about Example 1. The first is that in the vector space $C''(-\infty, \infty)$ of all twice differentiable functions defined on the entire real line, the two solutions $y_1 = e^x$ and $y_2 = e^{-x}$ are *linearly independent*. This means that the only solution of

$$C_1y_1 + C_2y_2 = 0$$

that is valid for all x is $C_1 = C_2 = 0$. The second observation is that every *linear combination* of y_1 and y_2 is also a solution of the linear differential equation. To see this, let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y &= C_1e^x + C_2e^{-x} \\ y' &= C_1e^x - C_2e^{-x} \\ y'' &= C_1e^x + C_2e^{-x}. \end{aligned}$$

Substituting into the differential equation $y'' - y = 0$ produces

$$y'' - y = (C_1e^x + C_2e^{-x}) - (C_1e^x + C_2e^{-x}) = 0.$$

So, $y = C_1e^x + C_2e^{-x}$ is a solution.

The next theorem, which is stated without proof, generalizes these observations.

REMARK

The solution

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

is the **general solution** of the differential equation.

Solutions of a Linear Homogeneous Differential Equation

Every n th-order linear homogeneous differential equation

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = 0$$

has n linearly independent solutions. Moreover, if $\{y_1, y_2, \dots, y_n\}$ is a set of linearly independent solutions, then every solution is of the form

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

where $C_1, C_2, \dots, C_n$ are real numbers.

In light of the preceding theorem, you can see the importance of being able to determine whether a set of solutions is linearly independent. Before describing a way of testing for linear independence, consider the definition below.

Definition of the Wronskian of a Set of Functions

Let $\{y_1, y_2, \dots, y_n\}$ be a set of functions, each of which has $n - 1$ derivatives on an interval I . The determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is the **Wronskian** of the set of functions.

REMARK

The Wronskian of a set of functions is named after the Polish mathematician Josef Maria Wronski (1778–1853).

EXAMPLE 2**Finding the Wronskian of a Set of Functions**

- a. The Wronskian of the set $\{1 - x, 1 + x, 2 - x\}$ is

$$W = \begin{vmatrix} 1 - x & 1 + x & 2 - x \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

- b. The Wronskian of the set $\{x, x^2, x^3\}$ is

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3.$$

The Wronskian in part (a) of Example 2 is **identically equal to zero**, because it is zero for any value of x . The Wronskian in part (b) is not identically equal to zero because values of x exist for which this Wronskian is nonzero.

The next theorem shows how the Wronskian of a set of functions can be used to test for linear independence.

REMARK

This test does *not* apply to an arbitrary set of functions. Each of the functions $y_1, y_2, \dots, y_n$ must be a solution of the same linear homogeneous differential equation of order n .

Wronskian Test for Linear Independence

Let $\{y_1, y_2, \dots, y_n\}$ be a set of n solutions of an n th-order linear homogeneous differential equation. This set is linearly independent if and only if the Wronskian is not identically equal to zero.

The proof of this theorem for the case where $n = 2$ is left as an exercise. (See Exercise 40.)

EXAMPLE 3**Testing a Set of Solutions for Linear Independence**

Determine whether $\{1, \cos x, \sin x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' + y' = 0.$$

SOLUTION

Begin by observing that each of the functions is a solution of $y''' + y' = 0$. (Check this.) Next, testing for linear independence produces the Wronskian of the three functions, as shown below.

$$\begin{aligned} W &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= \sin^2 x + \cos^2 x \\ &= 1 \end{aligned}$$

The Wronskian W is not identically equal to zero, so the set

$$\{1, \cos x, \sin x\}$$

is linearly independent. Moreover, this set consists of three linearly independent solutions of a third-order linear homogeneous differential equation, so the general solution is

$$y = C_1 + C_2 \cos x + C_3 \sin x$$

where C_1 , C_2 , and C_3 are real numbers. 

EXAMPLE 4**Testing a Set of Solutions for Linear Independence**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Determine whether $\{e^x, xe^x, (x+1)e^x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

SOLUTION

As in Example 3, begin by verifying that each of the functions is a solution of $y''' - 3y'' + 3y' - y = 0$. (This verification is left to you.) Testing for linear independence produces the Wronskian of the three functions, as shown below.

$$W = \begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix} = 0$$

So, the set $\{e^x, xe^x, (x+1)e^x\}$ is linearly dependent. 

In Example 4, the Wronskian is used to determine that the set $\{e^x, xe^x, (x+1)e^x\}$ is linearly dependent. Another way to determine the linear dependence of this set is to observe that the third function is a linear combination of the first two. That is,

$$(x+1)e^x = e^x + xe^x.$$

Verify that a different set, $\{e^x, xe^x, x^2e^x\}$, forms a linearly independent set of solutions of the differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

CONIC SECTIONS AND ROTATION

Every conic section in the xy -plane has an equation that can be written in the form

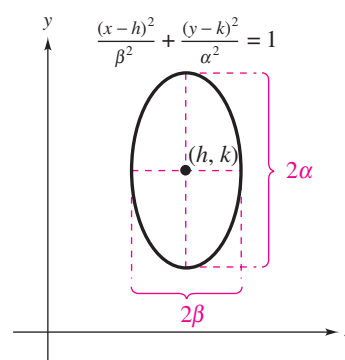
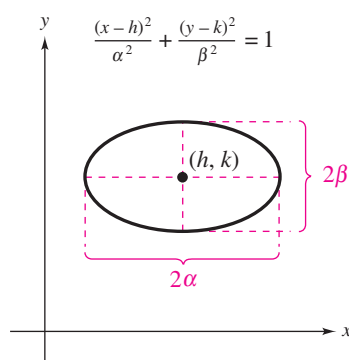
$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Identifying the graph of this equation is fairly simple as long as b , the coefficient of the xy -term, is zero. When b is zero, the conic axes are parallel to the coordinate axes, and the identification is accomplished by writing the equation in standard (completed square) form. The standard forms of the equations of the four basic conics are given in the summary below. For circles, ellipses, and hyperbolas, the point (h, k) is the center. For parabolas, the point (h, k) is the vertex.

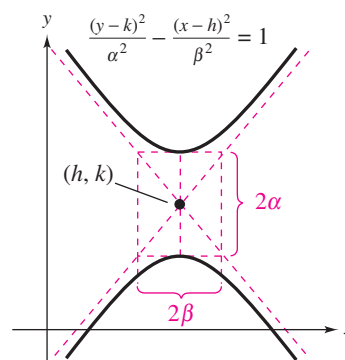
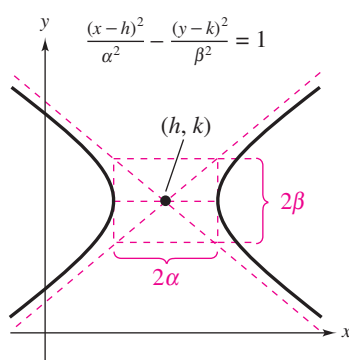
Standard Forms of Equations of Conics

Circle (r = radius): $(x - h)^2 + (y - k)^2 = r^2$

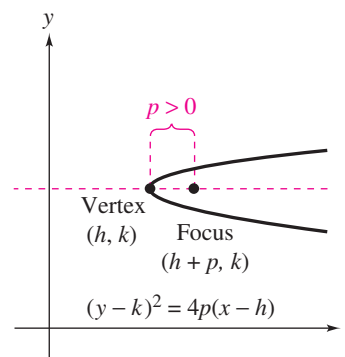
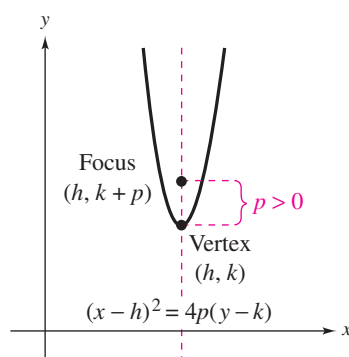
Ellipse (2α = major axis length, 2β = minor axis length):



Hyperbola (2α = transverse axis length, 2β = conjugate axis length):



Parabola (p = directed distance from vertex to focus):



EXAMPLE 5**Identifying Conic Sections**

- a. The standard form of $x^2 - 2x + 4y - 3 = 0$ is

$$(x - 1)^2 = 4(-1)(y - 1).$$

The graph of this equation is a parabola with the vertex at $(h, k) = (1, 1)$. The axis of the parabola is vertical. The directed distance p from the vertex to the focus is $p = -1$, so the focus is the point $(1, 0)$. Finally, the focus lies below the vertex, so the parabola opens downward, as shown in Figure 4.18(a).

- b. The standard form of $x^2 + 4y^2 + 6x - 8y + 9 = 0$ is

$$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} = 1.$$

The graph of this equation is an ellipse with its center at $(h, k) = (-3, 1)$. The major axis is horizontal, and its length is $2\alpha = 4$. The length of the minor axis is $2\beta = 2$. The vertices of this ellipse occur at $(-5, 1)$ and $(-1, 1)$, and the endpoints of the minor axis occur at $(-3, 2)$ and $(-3, 0)$, as shown in Figure 4.18(b).

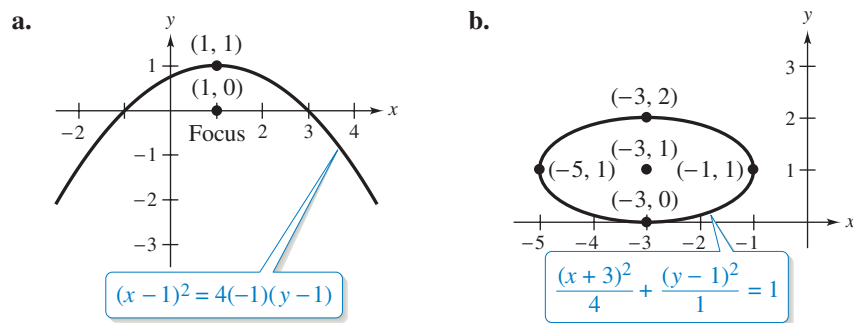


Figure 4.18

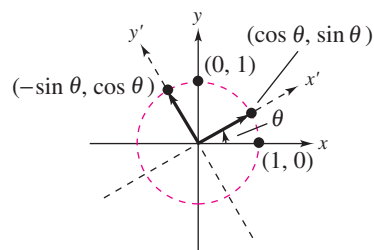
Note that the equations of the conics in Example 5 have no xy -term, so the axes of the graphs of these conics are parallel to the coordinate axes. For second-degree equations that have an xy -term, the axes of the graphs of the corresponding conics are not parallel to the coordinate axes. In such cases, it is helpful to *rotate* the standard axes to form a new x' -axis and y' -axis. The required rotation angle θ (measured counterclockwise) can be found using the equation $\cot 2\theta = (a - c)/b$. Then, the standard basis for R^2 ,

$$B = \{(1, 0), (0, 1)\}$$

rotates to form the new basis

$$B' = \{(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)\}$$

as shown below.



To find the coordinates of a point (x, y) relative to this new basis, you can use a transition matrix, as demonstrated in Example 6.

EXAMPLE 6**A Transition Matrix for Rotation in R^2**

Find the coordinates of a point (x, y) in R^2 relative to the basis

$$B' = \{(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)\}.$$

SOLUTION

By Theorem 4.21 you have

$$[B' \ B] = \begin{bmatrix} \cos \theta & -\sin \theta & 1 & 0 \\ \sin \theta & \cos \theta & 0 & 1 \end{bmatrix}.$$

B is the standard basis for R^2 , so P^{-1} is represented by $(B')^{-1}$. You can use the formula given in Section 2.3 (page 66) for the inverse of a 2×2 matrix to find $(B')^{-1}$. This results in

$$[I \ P^{-1}] = \begin{bmatrix} 1 & 0 & \cos \theta & \sin \theta \\ 0 & 1 & -\sin \theta & \cos \theta \end{bmatrix}.$$

By letting (x', y') be the coordinates of (x, y) relative to B' , you can use the transition matrix P^{-1} as shown below.

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

The x' - and y' -coordinates are $x' = x \cos \theta + y \sin \theta$ and $y' = -x \sin \theta + y \cos \theta$. 

The last two equations in Example 6 give the $x'y'$ -coordinates in terms of the xy -coordinates. To perform a rotation of axes for a general second-degree equation, it is helpful to express the xy -coordinates in terms of the $x'y'$ -coordinates. To do this, solve the last two equations in Example 6 for x and y to obtain

$$x = x' \cos \theta - y' \sin \theta \quad \text{and} \quad y = x' \sin \theta + y' \cos \theta.$$

Substituting these expressions for x and y into the given second-degree equation produces a second-degree equation in x' and y' that has no $x'y'$ -term.

Rotation of Axes

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

by rotating the coordinate axes counterclockwise through the angle θ , where θ is found using the equation $\cot 2\theta = \frac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions

$$x = x' \cos \theta - y' \sin \theta \quad \text{and} \quad y = x' \sin \theta + y' \cos \theta.$$

The proof of the above result is left to you. (See Exercise 80.)

**LINEAR ALGEBRA APPLIED**

A satellite dish is an antenna that is designed to transmit signals to or receive signals from a communications satellite. A standard satellite dish consists of a bowl-shaped surface and a *feed horn* that is aimed toward the surface. The bowl-shaped surface is typically in the shape of a rotated elliptic paraboloid. (See Section 7.4.) The cross section of the surface is typically in the shape of a rotated parabola.

Chris H. Galbraith/Shutterstock.com

Example 7 demonstrates how to identify the graph of a second-degree equation by rotating the coordinate axes.

EXAMPLE 7 Rotation of a Conic Section

Perform a rotation of axes to eliminate the xy -term in

$$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$$

and sketch the graph of the resulting equation in the $x'y'$ -plane.

SOLUTION

Find the angle of rotation θ using

$$\cot 2\theta = \frac{a - c}{b} = \frac{5 - 5}{-6} = 0.$$

This implies that $\theta = \pi/4$. So,

$$\sin \theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{2}}.$$

By substituting

$$x = x' \cos \theta - y' \sin \theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = x' \sin \theta + y' \cos \theta = \frac{1}{\sqrt{2}}(x' + y')$$

into the original equation and simplifying, verify that you obtain

$$(x')^2 + 4(y')^2 + 6x' - 8y' + 9 = 0.$$

Finally, by completing the square, the standard form of this equation is

$$\frac{(x' + 3)^2}{2^2} + \frac{(y' - 1)^2}{1^2} = \frac{(x' + 3)^2}{4} + \frac{(y' - 1)^2}{1} = 1$$

which is the equation of an ellipse, as shown in Figure 4.19.

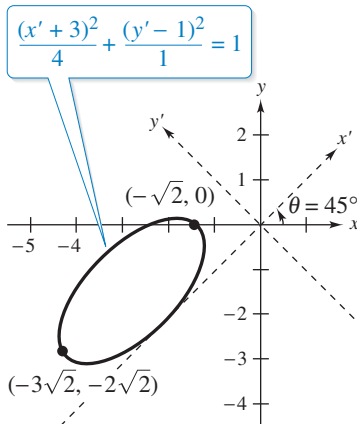


Figure 4.19

In Example 7, the new (rotated) basis for R^2 is

$$B' = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$$

and the coordinates of the vertices of the ellipse relative to B' are

$$\begin{bmatrix} -5 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

To find the coordinates of the vertices relative to the standard basis $B = \{(1, 0), (0, 1)\}$, use the equations

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = \frac{1}{\sqrt{2}}(x' + y')$$

to obtain $(-3\sqrt{2}, -2\sqrt{2})$ and $(-\sqrt{2}, 0)$, as shown in Figure 4.19.

4.8 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Determining Solutions of a Differential Equation In Exercises 1–12, determine which functions are solutions of the linear differential equation.

1. $y'' + y = 0$
 (a) e^x (b) $\sin x$
 (c) $\cos x$ (d) $\sin x - \cos x$
2. $y''' + y = 0$
 (a) e^x (b) e^{-x} (c) e^{-2x} (d) $2e^{-x}$
3. $y''' + y'' + y' + y = 0$
 (a) x (b) e^x (c) e^{-x} (d) xe^{-x}
4. $y'' - 6y' + 9y = 0$
 (a) e^{3x} (b) xe^{3x}
 (c) x^2e^{3x} (d) $(x + 3)e^{3x}$
5. $y^{(4)} + y''' - 2y'' = 0$
 (a) 1 (b) x (c) x^2 (d) e^x
6. $y^{(4)} - 16y = 0$
 (a) $3 \cos x$ (b) $3 \cos 2x$
 (c) e^{-2x} (d) $3e^{2x} - 4 \sin 2x$
7. $x^2y'' - 2y = 0$
 (a) $\frac{1}{x^2}$ (b) x^2 (c) e^{x^2} (d) e^{-x^2}
8. $y' + (2x - 1)y = 0$
 (a) e^{x-x^2} (b) $2e^{x-x^2}$ (c) $3e^{x-x^2}$ (d) $4e^{x-x^2}$
9. $xy' - 2y = 0$
 (a) $\sqrt{x}$ (b) x (c) x^2 (d) x^3
10. $xy'' + 2y' = 0$
 (a) x (b) $\frac{1}{x}$ (c) xe^x (d) xe^{-x}
11. $y'' - y' - 12y = 0$
 (a) e^{-4x} (b) e^{4x} (c) e^{-3x} (d) e^{3x}
12. $y' - 2xy = 0$
 (a) $3e^{x^2}$ (b) xe^{x^2} (c) x^2e^x (d) xe^{-x}

Finding the Wronskian for a Set of Functions In Exercises 13–26, find the Wronskian for the set of functions.

13. $\{x, -\sin x\}$ 14. $\{e^{3x}, \sin 2x\}$
15. $\{e^x, e^{-x}\}$ 16. $\{e^{x^2}, e^{-x^2}\}$
17. $\{x, \sin x, \cos x\}$ 18. $\{x, -\sin x, \cos x\}$
19. $\{e^{-x}, xe^{-x}, (x + 3)e^{-x}\}$ 20. $\{x, e^{-x}, e^x\}$
21. $\{1, e^x, e^{2x}\}$ 22. $\{x^2, e^{x^2}, x^2e^x\}$
23. $\{1, x, x^2, x^3\}$ 24. $\{x, x^2, e^x, e^{-x}\}$
25. $\{1, x, \cos x, e^{-x}\}$ 26. $\{x, e^x, \sin x, \cos x\}$

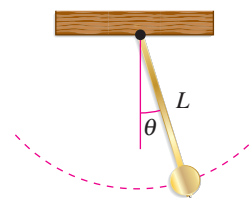
Showing Linear Independence In Exercises 27–30, show that the set of solutions of a second-order linear homogeneous differential equation is linearly independent.

27. $\{e^{ax}, e^{bx}\}$, $a \neq b$ 28. $\{e^{ax}, xe^{ax}\}$
29. $\{\cos ax, \sin ax\}$, $a \neq 0$
30. $\{e^{ax} \cos bx, e^{ax} \sin bx\}$, $b \neq 0$

Testing for Linear Independence In Exercises 31–38, (a) verify that each solution satisfies the differential equation, (b) test the set of solutions for linear independence, and (c) if the set is linearly independent, then write the general solution of the differential equation.

Differential Equation	Solutions
31. $y'' + 16y = 0$	$\{\sin 4x, \cos 4x\}$
32. $y'' - 4y' + 5y = 0$	$\{e^{2x} \sin x, e^{2x} \cos x\}$
33. $y''' + 4y'' + 4y' = 0$	$\{e^{-2x}, xe^{-2x}, (2x + 1)e^{-2x}\}$
34. $y''' + 4y' = 0$	$\{1, 2 \cos 2x, 2 + \cos 2x\}$
35. $y''' + 4y' = 0$	$\{1, \sin 2x, \cos 2x\}$
36. $y''' + 3y'' + 3y' + y = 0$	$\{e^{-x}, xe^{-x}, x^2e^{-x}\}$
37. $y''' + 3y'' + 3y' + y = 0$	$\{e^{-x}, xe^{-x}, e^{-x} + xe^{-x}\}$
38. $y^{(4)} - 2y''' + y'' = 0$	$\{1, x, e^x, xe^x\}$

39. **Pendulum** Consider a pendulum of length L that swings by the force of gravity only.



For small values of $\theta = \theta(t)$, the motion of the pendulum can be approximated by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where g is the acceleration due to gravity.

(a) Verify that

$$\left\{ \sin \sqrt{\frac{g}{L}}t, \cos \sqrt{\frac{g}{L}}t \right\}$$

is a set of linearly independent solutions of the differential equation.

(b) Find the general solution of the differential equation and show that it can be written in the form

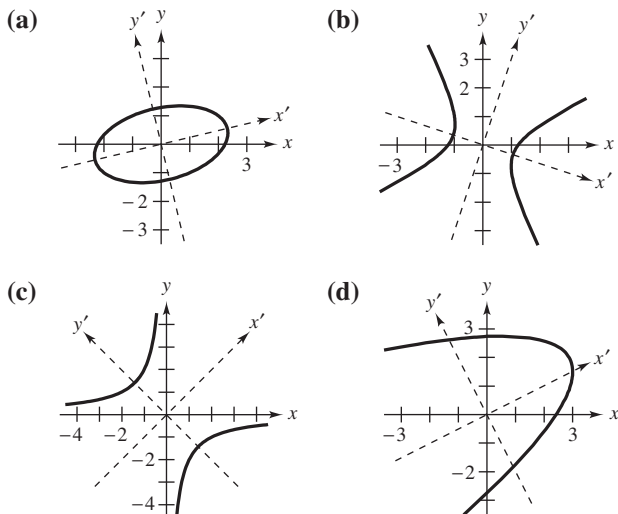
$$\theta(t) = A \cos \left[\sqrt{\frac{g}{L}}(t + \phi) \right].$$

- 40. Proof** Let $\{y_1, y_2\}$ be a set of solutions of a second-order linear homogeneous differential equation. Prove that this set is linearly independent if and only if the Wronskian is not identically equal to zero.
- 41. Writing** Is the sum of two solutions of a nonhomogeneous linear differential equation also a solution? Explain.
- 42. Writing** Is the scalar multiple of a solution of a nonhomogeneous linear differential equation also a solution? Explain.

Identifying and Graphing a Conic Section In Exercises 43–58, identify and sketch the graph of the conic section.

43. $y^2 + x = 0$ 44. $x^2 - 6y = 0$
 45. $x^2 + 4y^2 - 16 = 0$ 46. $5x^2 + 3y^2 - 15 = 0$
 47. $\frac{x^2}{9} - \frac{y^2}{16} - 1 = 0$ 48. $\frac{x^2}{36} - \frac{y^2}{49} = 1$
 49. $x^2 + 4x + 6y - 2 = 0$ 50. $y^2 - 6y - 4x + 21 = 0$
 51. $16x^2 + 36y^2 - 64x - 36y + 73 = 0$
 52. $4x^2 + y^2 - 8x + 3 = 0$
 53. $9x^2 - y^2 + 54x + 10y + 55 = 0$
 54. $4y^2 - 2x^2 - 4y - 8x - 15 = 0$
 55. $x^2 + 4y^2 + 4x + 32y + 64 = 0$
 56. $4y^2 + 4x^2 - 24x + 35 = 0$
 57. $2x^2 - y^2 + 4x + 10y - 22 = 0$
 58. $y^2 + 8x + 6y + 25 = 0$

Matching a Graph with an Equation In Exercises 59–62, match the graph with its equation. [The graphs are labeled (a), (b), (c), and (d).]



59. $xy + 2 = 0$
 60. $-2x^2 + 3xy + 2y^2 + 3 = 0$
 61. $x^2 - xy + 3y^2 - 5 = 0$
 62. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

Rotation of a Conic Section In Exercises 63–74, perform a rotation of axes to eliminate the xy -term, and sketch the graph of the conic.

63. $xy + 1 = 0$ 64. $xy - 8x - 4y = 0$
 65. $4x^2 + 2xy + 4y^2 - 15 = 0$
 66. $x^2 + 2xy + y^2 - 8x + 8y = 0$
 67. $2x^2 - 3xy - 2y^2 + 10 = 0$
 68. $5x^2 - 2xy + 5y^2 - 24 = 0$
 69. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
 70. $5x^2 - 6xy + 5y^2 - 12 = 0$
 71. $7x^2 - 6\sqrt{3}xy + 13y^2 - 64 = 0$
 72. $7x^2 - 2\sqrt{3}xy + 5y^2 = 16$
 73. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
 74. $x^2 + 2\sqrt{3}xy + 3y^2 - 2\sqrt{3}x + 2y + 16 = 0$

Rotation of a Degenerate Conic Section In Exercises 75–78, perform a rotation of axes to eliminate the xy -term, and sketch the graph of the “degenerate” conic.

75. $x^2 - 2xy + y^2 = 0$ 76. $5x^2 - 2xy + 5y^2 = 0$
 77. $x^2 + 2xy + y^2 - 1 = 0$ 78. $x^2 - 10xy + y^2 = 0$

- 79. Proof** Prove that a rotation of $\theta = \pi/4$ will eliminate the xy -term from the equation

$$ax^2 + bxy + ay^2 + dx + ey + f = 0.$$

- 80. Proof** Prove that a rotation of θ , where $\cot 2\theta = (a - c)/b$, will eliminate the xy -term from the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- 81. Proof** For the equation $ax^2 + bxy + cy^2 = 0$, define the matrix A as

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

- (a) Prove that if $|A| = 0$, then the graph of $ax^2 + bxy + cy^2 = 0$ is a line.
 (b) Prove that if $|A| \neq 0$, then the graph of $ax^2 + bxy + cy^2 = 0$ is two intersecting lines.

82. CAPSTONE

- (a) Explain how to use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence.
 (b) Explain how to eliminate the xy -term when it appears in the general equation of a conic section.

- 83.** Use your school's library, the Internet, or some other reference source to find real-life applications of (a) linear differential equations and (b) rotation of conic sections that are different than those discussed in this section.

4 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Vector Operations In Exercises 1–4, find (a) $\mathbf{u} + \mathbf{v}$, (b) $2\mathbf{v}$, (c) $\mathbf{u} - \mathbf{v}$, and (d) $3\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = (1, -2, -3)$, $\mathbf{v} = (3, 1, 0)$
2. $\mathbf{u} = (-1, 2, 1)$, $\mathbf{v} = (0, 1, 1)$
3. $\mathbf{u} = (3, -1, 2, 3)$, $\mathbf{v} = (0, 2, 2, 1)$
4. $\mathbf{u} = (0, 1, -1, 2)$, $\mathbf{v} = (1, 0, 0, 2)$

Solving a Vector Equation In Exercises 5–8, solve for $\mathbf{x}$, where $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (0, 2, 3)$, and $\mathbf{w} = (0, 1, 1)$.

5. $2\mathbf{x} - \mathbf{u} + 3\mathbf{v} + \mathbf{w} = \mathbf{0}$
6. $3\mathbf{x} + 2\mathbf{u} - \mathbf{v} + 2\mathbf{w} = \mathbf{0}$
7. $5\mathbf{u} - 2\mathbf{x} = 3\mathbf{v} + \mathbf{w}$
8. $3\mathbf{u} + 2\mathbf{x} = \mathbf{w} - \mathbf{v}$

Writing a Linear Combination In Exercises 9–12, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.

9. $\mathbf{v} = (3, 0, -6)$, $\mathbf{u}_1 = (1, -1, 2)$, $\mathbf{u}_2 = (2, 4, -2)$, $\mathbf{u}_3 = (1, 2, -4)$
10. $\mathbf{v} = (4, 4, 5)$, $\mathbf{u}_1 = (1, 2, 3)$, $\mathbf{u}_2 = (-2, 0, 1)$, $\mathbf{u}_3 = (1, 0, 0)$
11. $\mathbf{v} = (1, 2, 3, 5)$, $\mathbf{u}_1 = (1, 2, 3, 4)$, $\mathbf{u}_2 = (-1, -2, -3, 4)$, $\mathbf{u}_3 = (0, 0, 1, 1)$
12. $\mathbf{v} = (4, -13, -5, -4)$, $\mathbf{u}_1 = (1, -2, 1, 1)$, $\mathbf{u}_2 = (-1, 2, 3, 2)$, $\mathbf{u}_3 = (0, -1, -1, -1)$

Describing the Zero Vector and the Additive Inverse In Exercises 13–16, describe the zero vector and the additive inverse of a vector in the vector space.

13. $M_{4,2}$
14. P_8
15. R^5
16. $M_{2,3}$

Determining Subspaces In Exercises 17–24, determine whether W is a subspace of the vector space V .

17. $W = \{(x, y): x = 2y\}$, $V = R^2$
18. $W = \{(x, y): x - y = 1\}$, $V = R^2$
19. $W = \{(x, y): y = ax, a \text{ is an integer}\}$, $V = R^2$
20. $W = \{(x, y): y = ax^2\}$, $V = R^2$
21. $W = \{(x, 2x, 3x): x \text{ is a real number}\}$, $V = R^3$
22. $W = \{(x, y, z): x \geq 0\}$, $V = R^3$
23. $W = \{f: f(0) = -1\}$, $V = C[-1, 1]$
24. $W = \{f: f(-1) = 0\}$, $V = C[-1, 1]$

25. Which of the subsets of R^3 is a subspace of R^3 ?

- (a) $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 0\}$
- (b) $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 1\}$

26. Which of the subsets of R^3 is a subspace of R^3 ?

- (a) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$
- (b) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 1\}$

Spanning Sets, Linear Independence, and Bases In Exercises 27–32, determine whether the set (a) spans R^3 , (b) is linearly independent, and (c) is a basis for R^3 .

27. $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$
28. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$
29. $S = \{(-\frac{1}{2}, \frac{3}{4}, -1), (5, 2, 3), (-4, 6, -8)\}$
30. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$
31. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$
32. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

33. Determine whether

$$S = \{1 - t, 2t + 3t^2, t^2 - 2t^3, 2 + t^3\}$$

is a basis for P_3 .

34. Determine whether $S = \{1, t, 1 + t^2\}$ is a basis for P_2 .

Determining Whether a Set Is a Basis In Exercises 35 and 36, determine whether the set is a basis for $M_{2,2}$.

35. $S = \left\{ \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\}$
36. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

Finding the Nullspace, Nullity, and Rank of a Matrix In Exercises 37–42, find (a) the nullspace, (b) the nullity, and (c) the rank of the matrix A . Then verify that $\text{rank}(A) + \text{nullity}(A) = n$, where n is the number of columns of A .

$$37. A = \begin{bmatrix} -4 & 3 \\ 12 & -9 \end{bmatrix}$$

$$38. A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$39. A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$$

$$40. A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{bmatrix}$$

$$41. A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$$

$$42. A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 4 & 0 & 3 \\ -2 & 3 & 0 & 2 \\ 1 & 2 & 6 & 1 \end{bmatrix}$$

Finding a Basis for a Row Space and Rank In Exercises 43–46, find (a) a basis for the row space and (b) the rank of the matrix.

43. $\begin{bmatrix} 1 & 2 \\ -4 & 3 \\ 6 & 1 \end{bmatrix}$

44. $\begin{bmatrix} 2 & -1 & 4 \\ 1 & 5 & 6 \\ 1 & 16 & 14 \end{bmatrix}$

45. $\begin{bmatrix} 7 & 0 & 2 \\ 4 & 1 & 6 \\ -1 & 16 & 14 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Finding a Basis and Dimension In Exercises 47–50, find (a) a basis for and (b) the dimension of the solution space of the homogeneous system of linear equations.

47. $2x_1 + 4x_2 + 3x_3 - 6x_4 = 0$

$x_1 + 2x_2 + 2x_3 - 5x_4 = 0$

$3x_1 + 6x_2 + 5x_3 - 11x_4 = 0$

48. $16x_1 + 24x_2 + 8x_3 - 32x_4 = 0$

$4x_1 + 6x_2 + 2x_3 - 8x_4 = 0$

$2x_1 + 3x_2 + x_3 - 4x_4 = 0$

49. $x_1 - 3x_2 + x_3 + x_4 = 0$

$2x_1 + x_2 - x_3 + 2x_4 = 0$

$x_1 + 4x_2 - 2x_3 + x_4 = 0$

$5x_1 - 8x_2 + 2x_3 + 5x_4 = 0$

50. $-x_1 + 2x_2 - x_3 + 2x_4 = 0$

$-2x_1 + 2x_2 + x_3 + 4x_4 = 0$

$3x_1 + 2x_2 + 2x_3 + 5x_4 = 0$

$-3x_1 + 8x_2 + 5x_3 + 17x_4 = 0$

Finding a Coordinate Matrix In Exercises 51–56, given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis B for $\mathbb{R}^n$, find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

51. $B = \{(1, 1), (-1, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 3 & 5 \end{bmatrix}^T$

52. $B = \{(2, 0), (3, 3)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

53. $B = \{(\frac{1}{2}, \frac{1}{2}), (1, 0)\}$, $[\mathbf{x}]_B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T$

54. $B = \{(2, 4), (-1, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 4 & -7 \end{bmatrix}^T$

55. $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$,

$[\mathbf{x}]_B = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^T$

56. $B = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 4 & 0 & 2 \end{bmatrix}^T$

Finding a Coordinate Matrix In Exercises 57–62, find the coordinate matrix of $\mathbf{x}$ in $\mathbb{R}^n$ relative to the basis B' .

57. $B' = \{(5, 0), (0, -8)\}$, $\mathbf{x} = (2, 2)$

58. $B' = \{(2, 2), (0, -1)\}$, $\mathbf{x} = (-1, 2)$

59. $B' = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\}$, $\mathbf{x} = (3, -3, 0)$

60. $B' = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$, $\mathbf{x} = (4, -2, 9)$

61. $B' = \{(9, -3, 15, 4), (-3, 0, 0, -1), (0, -5, 6, 8), (-3, 4, -2, 3)\}$, $\mathbf{x} = (21, -5, 43, 14)$

62. $B' = \{(1, -1, 2, 1), (1, 1, -4, 3), (1, 2, 0, 3), (1, 2, -2, 0)\}$, $\mathbf{x} = (5, 3, -6, 2)$

Finding a Transition Matrix In Exercises 63–68, find the transition matrix from B to B' .

63. $B = \{(1, -1), (3, 1)\}$, $B' = \{(1, 0), (0, 1)\}$

64. $B = \{(1, -1), (3, 1)\}$, $B' = \{(1, 2), (-1, 0)\}$

65. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

$B' = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

66. $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$,

$B' = \{(1, 2, 3), (0, 1, 0), (1, 0, 1)\}$

67. $B = \{(1, 1, 2), (2, 3, 4), (3, 3, 3)\}$,

$B' = \{(7, -1, -1), (-3, 1, 0), (-3, 0, 1)\}$

68. $B = \{(1, 1, 1), (3, 4, 3), (3, 3, 4)\}$,

$B' = \{(1, -1, \frac{2}{3}), (-2, 1, 0), (1, 0, -\frac{1}{3})\}$

Finding Transition and Coordinate Matrices In Exercises 69–72, (a) find the transition matrix from B to B' , (b) find the transition matrix from B' to B , (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[\mathbf{x}]_{B'}$, given the coordinate matrix $[\mathbf{x}]_B$.

69. $B = \{(-2, 1), (1, -1)\}$, $B' = \{(0, 2), (1, 1)\}$,

$[\mathbf{x}]_B = \begin{bmatrix} 6 & -6 \end{bmatrix}^T$

70. $B = \{(1, 0), (1, -1)\}$, $B' = \{(1, 1), (1, -1)\}$,

$[\mathbf{x}]_B = \begin{bmatrix} 2 & -2 \end{bmatrix}^T$

71. $B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$,

$B' = \{(0, 0, 1), (0, 1, 1), (1, 1, 1)\}$,

$[\mathbf{x}]_B = \begin{bmatrix} -1 & 2 & -3 \end{bmatrix}^T$

72. $B = \{(1, 1, -1), (1, 1, 0), (1, -1, 0)\}$,

$B' = \{(1, -1, 2), (2, 2, -1), (2, 2, 2)\}$,

$[\mathbf{x}]_B = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}^T$

73. Let W be the subspace of P_3 [the set of all polynomials $p(x)$ of degree 3 or less] such that $p(0) = 0$, and let U be the subspace of P_3 such that $p(1) = 0$. Find a basis for W , a basis for U , and a basis for their intersection $W \cap U$.

74. **Calculus** Let $V = C'(-\infty, \infty)$, the vector space of all continuously differentiable functions on the real line.

(a) Prove that $W = \{f: f' = 4f\}$ is a subspace of V .

(b) Prove that $U = \{f: f' = f + 1\}$ is not a subspace of V .

75. **Writing** Let $B = \{p_1(x), p_2(x), \dots, p_n(x), p_{n+1}(x)\}$ be a basis for P_n . Must B contain a polynomial of each degree 0, 1, 2, $\dots$, n ? Explain.

76. **Proof** Let A and B be $n \times n$ square matrices with $A \neq O$ and $B \neq O$. Prove that if A is symmetric and B is skew-symmetric ($B^T = -B$), then $\{A, B\}$ is a linearly independent set.

77. **Proof** Let $V = P_5$ and consider the set W of all polynomials of the form $(x^3 + x)p(x)$, where $p(x)$ is in P_2 . Is W a subspace of V ? Prove your answer.

78. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be three linearly independent vectors in a vector space V . Is the set $\{\mathbf{v}_1 - 2\mathbf{v}_2, 2\mathbf{v}_2 - 3\mathbf{v}_3, 3\mathbf{v}_3 - \mathbf{v}_1\}$ linearly dependent or linearly independent? Explain.

79. **Proof** Let A be an $n \times n$ square matrix. Prove that the row vectors of A are linearly dependent if and only if the column vectors of A are linearly dependent.

80. **Proof** Let A be an $n \times n$ square matrix, and let λ be a scalar. Prove that the set

$$S = \{\mathbf{x}: A\mathbf{x} = \lambda\mathbf{x}\}$$

is a subspace of R^n . Determine the dimension of S when $\lambda = 3$ and

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

81. Let $f(x) = x$ and $g(x) = |x|$.

(a) Show that f and g are linearly independent in $C[-1, 1]$.

(b) Show that f and g are linearly dependent in $C[0, 1]$.

82. Describe how the domain of a set of functions can influence whether the set is linearly independent or dependent.

True or False? In Exercises 83–86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

83. (a) The standard operations in R^n are vector addition and scalar multiplication.

(b) The additive inverse of a vector is not unique.

(c) A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

84. (a) The set $W = \{(0, x^2, x^3): x^2 \text{ and } x^3 \text{ are real numbers}\}$ is a subspace of R^3 .

(b) A set of vectors S in a vector space V is a basis for V when S spans V and S is linearly independent.

(c) If A is an invertible $n \times n$ matrix, then the n row vectors of A are linearly dependent.

85. (a) The set of all n -tuples is n -space and is denoted by R^n .

(b) The additive identity of a vector space is not unique.

(c) Once a theorem has been proved for an abstract vector space, you need not give separate proofs for n -tuples, matrices, and polynomials.

86. (a) The set of points on the line $x + y = 0$ is a subspace of R^2 .

(b) Elementary row operations preserve the column space of the matrix A .

Determining Solutions of a Differential Equation In Exercises 87–90, determine which functions are solutions of the linear differential equation.

87. $y'' - y' - 6y = 0$

(a) e^{3x} (b) e^{2x} (c) e^{-3x} (d) e^{-2x}

88. $y^{(4)} - y = 0$

(a) e^x (b) e^{-x} (c) $\cos x$ (d) $\sin x$

89. $y' + 2y = 0$

(a) e^{-2x} (b) xe^{-2x} (c) x^2e^{-x} (d) $2xe^{-2x}$

90. $y'' + 25y = 0$

(a) $\sin 5x + \cos 5x$ (b) $5 \sin x + 5 \cos x$

(c) $\sin 5x$ (d) $\cos 5x$

Finding the Wronskian for a Set of Functions In Exercises 91–94, find the Wronskian for the set of functions.

91. $\{1, x, e^x\}$

92. $\{2, x^2, 3 + x\}$

93. $\{1, \sin 2x, \cos 2x\}$

94. $\{x, \sin^2 x, \cos^2 x\}$

Testing for Linear Independence In Exercises 95–98, (a) verify that each solution satisfies the differential equation, (b) test the set of solutions for linear independence, and (c) if the set is linearly independent, then write the general solution of the differential equation.

Differential Equation

Solutions

95. $y'' + 6y' + 9y = 0$

$\{e^{-3x}, xe^{-3x}\}$

96. $y'' + 6y' + 9y = 0$

$\{e^{-3x}, 3e^{-3x}\}$

97. $y''' - 6y'' + 11y' - 6y = 0$

$\{e^x, e^{2x}, e^x - e^{2x}\}$

98. $y'' + 9y = 0$

$\{\sin 3x, \cos 3x\}$

Identifying and Graphing a Conic Section In Exercises 99–106, identify and sketch the graph of the conic section.

99. $x^2 + y^2 + 4x - 2y - 11 = 0$

100. $9x^2 + 9y^2 + 18x - 18y + 14 = 0$

101. $x^2 - y^2 + 2x - 3 = 0$

102. $4x^2 - y^2 + 8x - 6y + 4 = 0$

103. $2x^2 - 20x - y + 46 = 0$

104. $y^2 - 4x - 4 = 0$

105. $4x^2 + y^2 + 32x + 4y + 63 = 0$

106. $16x^2 + 25y^2 - 32x - 50y + 16 = 0$

Rotation of a Conic Section In Exercises 107–110, perform a rotation of axes to eliminate the xy -term, and sketch the graph of the conic.

107. $xy = 3$

108. $9x^2 + 4xy + 9y^2 - 20 = 0$

109. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

110. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0$

4 Projects

1 Solutions of Linear Systems

Write a paragraph to answer the question. Do not perform any calculations, but instead base your explanations on appropriate properties from the text.

1. One solution of the homogeneous linear system

$$x + 2y + z + 3w = 0$$

$$x - y + w = 0$$

$$y - z + 2w = 0$$

is $x = -2, y = -1, z = 1$, and $w = 1$. Explain why $x = 4, y = 2, z = -2$, and $w = -2$ is also a solution.

2. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Explain why the vector $2\mathbf{x}_1 - 3\mathbf{x}_2$ is also a solution.

3. Consider the two systems represented by the augmented matrices.

$$\left[\begin{array}{cccc} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array} \right] \quad \left[\begin{array}{cccc} 1 & 1 & -5 & -9 \\ 1 & 0 & -2 & -3 \\ 2 & -1 & -1 & 0 \end{array} \right]$$

If the first system is consistent, then why is the second system also consistent?

4. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the linear system $A\mathbf{x} = \mathbf{b}$. Is the vector $2\mathbf{x}_1 - 3\mathbf{x}_2$ also a solution? Why or why not?
5. The linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ are consistent. Is the system $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$ necessarily consistent? Why or why not?

2 Direct Sum

In this project, you will explore the **sum** and **direct sum** of subspaces. In Exercise 58 in Section 4.3, you proved that for two subspaces U and W of a vector space V , the sum $U + W$ of the subspaces, defined as $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$, is also a subspace of V .

1. Consider the subspaces of $V = \mathbb{R}^3$ below.

$$U = \{(x, y, x - y) : x, y \in \mathbb{R}\}$$

$$W = \{(x, 0, x) : x \in \mathbb{R}\}$$

$$Z = \{(x, x, x) : x \in \mathbb{R}\}$$

Find $U + W$, $U + Z$, and $W + Z$.

2. If U and W are subspaces of V such that $V = U + W$ and $U \cap W = \{\mathbf{0}\}$, then prove that every vector in V has a *unique* representation of the form $\mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ is in U and $\mathbf{w}$ is in W . V is called the **direct sum** of U and W , and is written as $V = U \oplus W$. Direct sum

Which of the sums in part (1) are direct sums?

3. Let $V = U \oplus W$, and let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ be a basis for the subspace U and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ be a basis for the subspace W . Prove that the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{w}_1, \dots, \mathbf{w}_m\}$ is a basis for V .
4. Consider the subspaces $U = \{(x, 0, y) : x, y \in \mathbb{R}\}$ and $W = \{(0, x, y) : x, y \in \mathbb{R}\}$ of $V = \mathbb{R}^3$. Show that $\mathbb{R}^3 = U + W$. Is $\mathbb{R}^3$ the *direct sum* of U and W ? What are the dimensions of U , W , $U \cap W$, and $U + W$? Formulate a conjecture that relates the dimensions of U , W , $U \cap W$, and $U + W$.
5. Do there exist two two-dimensional subspaces of $\mathbb{R}^3$ whose intersection is the zero vector? Why or why not?

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