

Elementary Linear Algebra

8e

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Australia • Brazil • Mexico • Singapore • United Kingdom • United States

Elementary Linear Algebra
Eighth Edition**Ron Larson**

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A1

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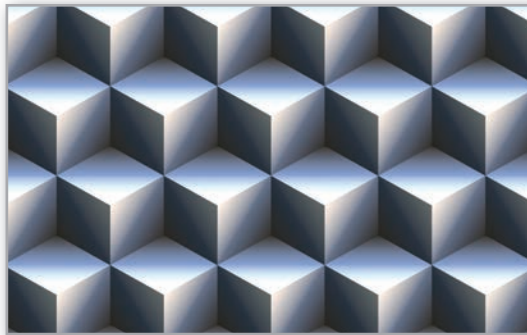
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Technology Guide*

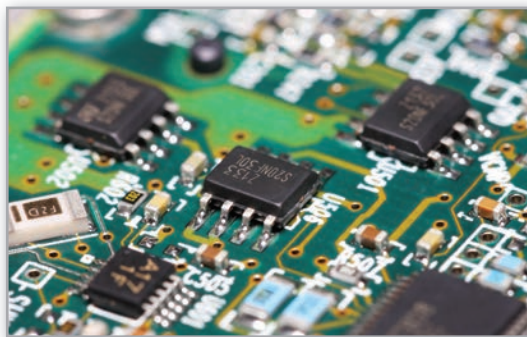
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6 Linear Transformations

- 6.1 Introduction to Linear Transformations
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6.1 Introduction to Linear Transformations

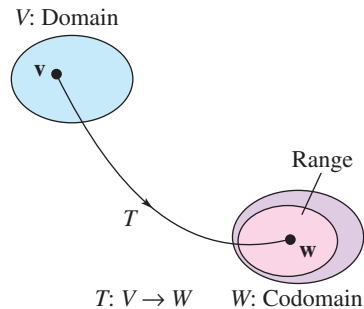
-  Find the image and preimage of a function.
-  Show that a function is a linear transformation, and find a linear transformation.

IMAGES AND PREIMAGES OF FUNCTIONS

In this chapter, you will learn about functions that **map** a vector space V into a vector space W . This type of function is denoted by

$$T: V \rightarrow W.$$

The standard function terminology is used for such functions. For instance, V is the **domain** of T , and W is the **codomain** of T . If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that $T(\mathbf{v}) = \mathbf{w}$, then $\mathbf{w}$ is the **image** of $\mathbf{v}$ under T . The set of all images of vectors in V is the **range** of T , and the set of all $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$ is the **preimage** of $\mathbf{w}$. (See below.)



REMARK

For a vector

$$\mathbf{v} = (v_1, v_2, \dots, v_n)$$

in R^n , it would be more correct to use double parentheses to denote $T(\mathbf{v})$ as $T(\mathbf{v}) = T((v_1, v_2, \dots, v_n))$. For convenience, however, drop one set of parentheses to produce

$$T(\mathbf{v}) = T(v_1, v_2, \dots, v_n).$$

EXAMPLE 1

A Function from R^2 into R^2

See LarsonLinearAlgebra.com for an interactive version of this type of example.

For any vector $\mathbf{v} = (v_1, v_2)$ in R^2 , define $T: R^2 \rightarrow R^2$ by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2).$$

- a. Find the image of $\mathbf{v} = (-1, 2)$.
- b. Find the image of $\mathbf{v} = (0, 0)$.
- c. Find the preimage of $\mathbf{w} = (-1, 11)$.

SOLUTION

- a. For $\mathbf{v} = (-1, 2)$, you have

$$T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3).$$

- b. If $\mathbf{v} = (0, 0)$, then

$$T(0, 0) = (0 - 0, 0 + 2(0)) = (0, 0).$$

- c. If $T(\mathbf{v}) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$, then

$$\begin{aligned} v_1 - v_2 &= -1 \\ v_1 + 2v_2 &= 11. \end{aligned}$$

This system of equations has the unique solution $v_1 = 3$ and $v_2 = 4$. So, the preimage of $(-1, 11)$ is the set in R^2 consisting of the single vector $(3, 4)$. 

LINEAR TRANSFORMATIONS

This chapter centers on functions that map one vector space into another and preserve the operations of vector addition and scalar multiplication. Such functions are called **linear transformations**.

Definition of a Linear Transformation

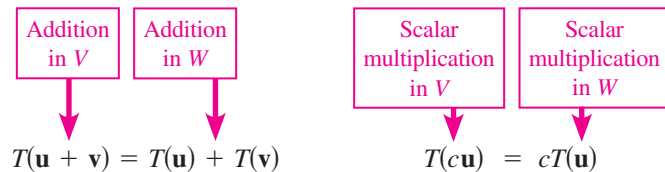
Let V and W be vector spaces. The function

$$T: V \rightarrow W$$

is a **linear transformation** of V into W when the two properties below are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and for any scalar c .

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

A linear transformation is *operation preserving* because the same result occurs whether you perform the operations of addition and scalar multiplication before or after applying the linear transformation. Although the same symbols denote the vector operations in both V and W , you should note that the operations may be different, as shown in the diagram below.



EXAMPLE 2

Verifying a Linear Transformation from R^2 into R^2

Show that the function in Example 1 is a linear transformation from R^2 into R^2 .

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

SOLUTION

To show that the function T is a linear transformation, you must show that it preserves vector addition and scalar multiplication. To do this, let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ be vectors in R^2 and let c be any real number. Then, using the properties of vector addition and scalar multiplication, you have the two statements below.

1. $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$, so you have

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

2. $c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$, so you have

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) \\ &= (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}). \end{aligned}$$

So, T is a linear transformation.

REMARK

A linear transformation $T: V \rightarrow V$ from a vector space into itself (as in Example 2) is called a **linear operator**.

Many common functions are not linear transformations, as demonstrated in Example 3.

EXAMPLE 3**Some Functions That Are Not Linear Transformations**

- a. $f(x) = \sin x$ is not a linear transformation from R into R because, in general, $\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$. For example,

$$\sin[(\pi/2) + (\pi/3)] \neq \sin(\pi/2) + \sin(\pi/3).$$

- b. $f(x) = x^2$ is not a linear transformation from R into R because, in general, $(x_1 + x_2)^2 \neq x_1^2 + x_2^2$. For example, $(1 + 2)^2 \neq 1^2 + 2^2$.

- c. $f(x) = x + 1$ is not a linear transformation from R into R because

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

whereas

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2.$$

$$\text{So } f(x_1 + x_2) \neq f(x_1) + f(x_2).$$

REMARK

The function in Example 3(c) suggests two uses of the term *linear*. The function $f(x) = x + 1$ is a linear function because its graph is a line. It is not a linear transformation from the vector space R into R , however, because it does not preserve vector addition or scalar multiplication.

REMARK

One advantage of Theorem 6.1 is that it provides a quick way to identify functions that are not linear transformations. That is, all four conditions of the theorem must be true of a linear transformation, so it follows that if any one of the properties is not satisfied for a function T , then the function is not a linear transformation. For example, the function

$$T(x_1, x_2) = (x_1 + 1, x_2)$$

is not a linear transformation from R^2 into R^2 because $T(0, 0) \neq (0, 0)$.

Two simple linear transformations are the **zero transformation** and the **identity transformation**, which are defined below.

1. $T(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v}$ **Zero transformation** ($T: V \rightarrow W$)

2. $T(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v}$ **Identity transformation** ($T: V \rightarrow V$)

You are asked to prove that these are linear transformations in Exercise 77.

Note that the linear transformation in Example 1 has the property that the zero vector maps to itself. That is, $T(\mathbf{0}) = \mathbf{0}$, as shown in Example 1(b). This property is true for all linear transformations, as stated in the first property of the theorem below.

THEOREM 6.1 Properties of Linear Transformations

Let T be a linear transformation from V into W , where $\mathbf{u}$ and $\mathbf{v}$ are in V . Then the properties listed below are true.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n).$$

PROOF

To prove the first property, note that $0\mathbf{v} = \mathbf{0}$. Then it follows that

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}.$$

The second property follows from $-\mathbf{v} = (-1)\mathbf{v}$, which implies that

$$T(-\mathbf{v}) = T[(-1)\mathbf{v}] = (-1)T(\mathbf{v}) = -T(\mathbf{v}).$$

The third property follows from $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, which implies that

$$T(\mathbf{u} - \mathbf{v}) = T[\mathbf{u} + (-1)\mathbf{v}] = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}).$$

The proof of the fourth property is left to you.

Property 4 of Theorem 6.1 suggests that a linear transformation $T: V \rightarrow W$ is determined completely by its action on a basis for V . In other words, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for the vector space V and if $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$ are given, then $T(\mathbf{v})$ can be determined for *any* $\mathbf{v}$ in V . Example 4 demonstrates the use of this property.

EXAMPLE 4**Linear Transformations and Bases**

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1, 0, 0) = (2, -1, 4)$$

$$T(0, 1, 0) = (1, 5, -2)$$

$$T(0, 0, 1) = (0, 3, 1).$$

Find $T(2, 3, -2)$.

SOLUTION

$(2, 3, -2) = 2(1, 0, 0) + 3(0, 1, 0) - 2(0, 0, 1)$, so use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(2, 3, -2) &= 2T(1, 0, 0) + 3T(0, 1, 0) - 2T(0, 0, 1) \\ &= 2(2, -1, 4) + 3(1, 5, -2) - 2(0, 3, 1) \\ &= (7, 7, 0). \end{aligned}$$

In the next example, a matrix defines a linear transformation from R^2 into R^3 . The vector $\mathbf{v} = (v_1, v_2)$ is in the matrix form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

so it can be multiplied *on the left* by a matrix of size 3×2 .

EXAMPLE 5**A Linear Transformation Defined by a Matrix**

Define the function $T: R^2 \rightarrow R^3$ as

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

- Find $T(\mathbf{v})$ when $\mathbf{v} = (2, -1)$.
- Show that T is a linear transformation from R^2 into R^3 .

SOLUTION

- $\mathbf{v} = (2, -1)$, so you have

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$



Vector
in R^2



Vector
in R^3

which means that $T(2, -1) = (6, 3, 0)$.

- Begin by observing that T maps a vector in R^2 to a vector in R^3 . To show that T is a linear transformation, use properties given in Theorem 2.3. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in R^2 , the distributive property of matrix multiplication over addition produces

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}).$$

Similarly, for any vector $\mathbf{u}$ in R^2 and any scalar c , the commutative property of scalar multiplication with matrix multiplication produces

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u}).$$

Example 5 illustrates an important result regarding the representation of linear transformations from R^n into R^m . This result is presented in two stages. Theorem 6.2 below states that every $m \times n$ matrix represents a linear transformation from R^n into R^m . Then, in Section 6.3, you will see the converse—that every linear transformation from R^n into R^m can be represented by an $m \times n$ matrix.

Note that the solution of Example 5(b) makes no reference specifically to the matrix A that defines T . So, this solution serves as a general proof that the function defined by any $m \times n$ matrix is a linear transformation from R^n into R^m .

REMARK

The $m \times n$ zero matrix corresponds to the zero transformation from R^n into R^m , and the $n \times n$ identity matrix I_n corresponds to the identity transformation from R^n into R^n .

THEOREM 6.2 Linear Transformation Given by a Matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m . In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in R^n and $m \times 1$ matrices represent the vectors in R^m .

Be sure you see that an $m \times n$ matrix A defines a linear transformation from R^n into R^m :

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$



Vector
in R^n



Vector
in R^m

EXAMPLE 6

Linear Transformations Given by Matrices

Consider the linear transformation $T: R^n \rightarrow R^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m for the linear transformation represented by each matrix.

a. $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$

SOLUTION

a. The size of this matrix is 3×3 , so it defines a linear transformation from R^3 into R^3 .

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$



Vector
in R^3



Vector
in R^3

b. The size of this matrix is 3×2 , so it defines a linear transformation from R^2 into R^3 .

c. The size of this matrix is 2×4 , so it defines a linear transformation from R^4 into R^2 .

The next example discusses a common type of linear transformation from R^2 into R^2 .

EXAMPLE 7

Rotation in R^2

Show that the linear transformation $T: R^2 \rightarrow R^2$ represented by the matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ .

SOLUTION

From Theorem 6.2, you know that T is a linear transformation. To show that it rotates every vector in R^2 counterclockwise through the angle θ , let $\mathbf{v} = (x, y)$ be a vector in R^2 . Using polar coordinates, you can write $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v} &= (x, y) \\ &= (r \cos \alpha, r \sin \alpha) \end{aligned}$$

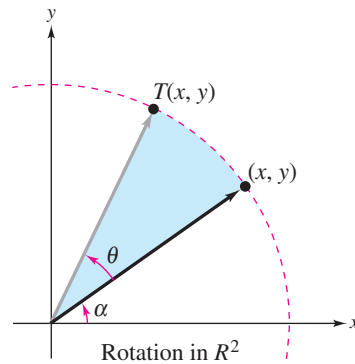
where r is the length of $\mathbf{v}$ and α is the angle from the positive x -axis counterclockwise to the vector $\mathbf{v}$. Now, applying the linear transformation T to $\mathbf{v}$ produces

$$\begin{aligned} T(\mathbf{v}) &= A\mathbf{v} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} r \cos \alpha \\ r \sin \alpha \end{bmatrix} \\ &= \begin{bmatrix} r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \\ r \sin \theta \cos \alpha + r \cos \theta \sin \alpha \end{bmatrix} \\ &= \begin{bmatrix} r \cos(\theta + \alpha) \\ r \sin(\theta + \alpha) \end{bmatrix}. \end{aligned}$$

Verify that the vector $T(\mathbf{v})$ has the same length as $\mathbf{v}$. Furthermore, the angle from the positive x -axis to $T(\mathbf{v})$ is

$$\theta + \alpha$$

so $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ , as shown below.



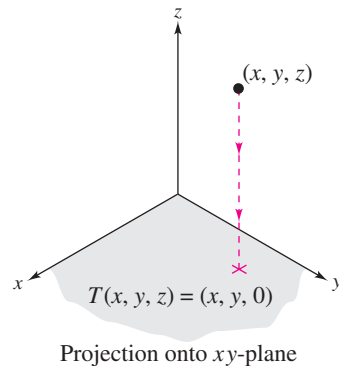
The linear transformation in Example 7 is a **rotation** in R^2 . Rotations in R^2 preserve both vector length and the angle between two vectors. That is, $\|T(\mathbf{u})\| = \|\mathbf{u}\|$, $\|T(\mathbf{v})\| = \|\mathbf{v}\|$, and the angle between $T(\mathbf{u})$ and $T(\mathbf{v})$ is equal to the angle between $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 8**A Projection in R^3**

The linear transformation $T: R^3 \rightarrow R^3$ represented by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a **projection** in R^3 . If $\mathbf{v} = (x, y, z)$ is a vector in R^3 , then $T(\mathbf{v}) = (x, y, 0)$. In other words, T maps every vector in R^3 to its orthogonal projection in the xy -plane, as shown below.



So far, only linear transformations from R^n into R^m or from R^n into R^n have been discussed. The remainder of this section considers some linear transformations involving vector spaces other than R^n .

EXAMPLE 9**A Linear Transformation from $M_{m,n}$ into $M_{n,m}$**

Let $T: M_{m,n} \rightarrow M_{n,m}$ be the function that maps an $m \times n$ matrix A to its transpose. That is,

$$T(A) = A^T.$$

Show that T is a linear transformation.

SOLUTION

Let A and B be $m \times n$ matrices and let c be a scalar. From Theorem 2.6 you have

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$$

and

$$T(cA) = (cA)^T = c(A^T) = cT(A).$$

So, T is a linear transformation from $M_{m,n}$ into $M_{n,m}$.

**LINEAR ALGEBRA APPLIED**

Many multivariate statistical methods can use linear transformations. For instance, in a *multiple regression analysis*, there are two or more independent variables and a single dependent variable. A linear transformation is useful for finding weights to be assigned to the independent variables to predict the value of the dependent variable. Also, in a *canonical correlation analysis*, there are two or more independent variables and two or more dependent variables. Linear transformations can help find a linear combination of the independent variables to predict the value of a linear combination of the dependent variables.

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EXAMPLE 10**The Differential Operator (Calculus)**

Let $C'[a, b]$ be the set of all functions whose derivatives are continuous on $[a, b]$. Show that the differential operator D_x defines a linear transformation from $C'[a, b]$ into $C[a, b]$.

SOLUTION

Using operator notation, you can write

$$D_x(f) = \frac{d}{dx}[f]$$

where f is in $C'[a, b]$. To show that D_x is a linear transformation, you must use calculus. Specifically, the derivative of the sum of two differentiable functions is equal to the sum of their derivatives, so you have

$$D_x(f + g) = \frac{d}{dx}[f + g] = \frac{d}{dx}[f] + \frac{d}{dx}[g] = D_x(f) + D_x(g)$$

where g is also in $C'[a, b]$. Similarly, the derivative of a scalar multiple cf of a differentiable function is equal to the scalar multiple of the derivative, so you have

$$D_x(cf) = \frac{d}{dx}[cf] = c\left(\frac{d}{dx}[f]\right) = cD_x(f).$$

The sum of two continuous functions is continuous, and the scalar multiple of a continuous function is continuous, so D_x is a linear transformation from $C'[a, b]$ into $C[a, b]$. 

The linear transformation D_x in Example 10 is called the **differential operator**. For polynomials, the differential operator is a linear transformation from P_n into P_{n-1} because the derivative of a polynomial function of degree $n \geq 1$ is a polynomial function of degree $n - 1$. That is,

$$D_x(a_0 + a_1x + \cdots + a_nx^n) = a_1 + \cdots + na_nx^{n-1}.$$

The next example describes a linear transformation from the vector space of polynomial functions P into the vector space of real numbers R .

EXAMPLE 11**The Definite Integral as a Linear Transformation (Calculus)**

Consider $T: P \rightarrow R$ defined by

$$T(p) = \int_a^b p(x) dx$$

where p is a polynomial function. Show that T is a linear transformation from P , the vector space of polynomial functions, into R , the vector space of real numbers.

SOLUTION

Using properties of definite integrals, you can write

$$T(p + q) = \int_a^b [p(x) + q(x)] dx = \int_a^b p(x) dx + \int_a^b q(x) dx = T(p) + T(q)$$

where q is a polynomial function, and

$$T(cp) = \int_a^b [cp(x)] dx = c \int_a^b p(x) dx = cT(p)$$

where c is a scalar. So, T is a linear transformation. 

6.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding an Image and a Preimage In Exercises 1–8, use the function to find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$.

- $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$,
 $\mathbf{v} = (3, -4)$, $\mathbf{w} = (3, 19)$
- $T(v_1, v_2) = (v_1, 2v_2 - v_1, v_2)$,
 $\mathbf{v} = (0, 4)$, $\mathbf{w} = (2, 4, 3)$
- $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$,
 $\mathbf{v} = (-4, 5, 1)$, $\mathbf{w} = (4, 1, -1)$
- $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$,
 $\mathbf{v} = (2, 3, 0)$, $\mathbf{w} = (-11, -1, 10)$
- $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2)$,
 $\mathbf{v} = (2, -3, -1)$, $\mathbf{w} = (3, 9)$
- $T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2)$,
 $\mathbf{v} = (2, 1, 4)$, $\mathbf{w} = (-1, 2)$
- $T(v_1, v_2) = \left(\frac{\sqrt{2}}{2}v_1 - \frac{\sqrt{2}}{2}v_2, v_1 + v_2, 2v_1 - v_2\right)$,
 $\mathbf{v} = (1, 1)$, $\mathbf{w} = (-5\sqrt{2}, -2, -16)$
- $T(v_1, v_2) = \left(\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2, v_1 - v_2, v_2\right)$,
 $\mathbf{v} = (2, 4)$, $\mathbf{w} = (\sqrt{3}, 2, 0)$

Linear Transformations In Exercises 9–22, determine whether the function is a linear transformation.

- $T: R^2 \rightarrow R^2$, $T(x, y) = (x, 1)$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (x, y^2)$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + y, x - y, z)$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + 1, y + 1, z + 1)$
- $T: R^2 \rightarrow R^3$, $T(x, y) = (\sqrt{x}, xy, \sqrt{y})$
- $T: R^2 \rightarrow R^3$, $T(x, y) = (x^2, xy, y^2)$
- $T: M_{2,2} \rightarrow R$, $T(A) = |A|$
- $T: M_{2,2} \rightarrow R$, $T(A) = a + b + c + d$, where
 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- $T: M_{2,2} \rightarrow R$, $T(A) = a - b - c - d$, where
 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- $T: M_{2,2} \rightarrow R$, $T(A) = b^2$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- $T: M_{3,3} \rightarrow M_{3,3}$, $T(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$

$$20. T: M_{3,3} \rightarrow M_{3,3}, T(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix} A$$

- $T: P_2 \rightarrow P_2$, $T(a_0 + a_1x + a_2x^2) = (a_0 + a_1 + a_2) + (a_1 + a_2)x + a_2x^2$
- $T: P_2 \rightarrow P_2$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$

- Let T be a linear transformation from R^2 into R^2 such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 1)$. Find $T(1, 4)$ and $T(-2, 1)$.
- Let T be a linear transformation from R^2 into R^2 such that $T(1, 2) = (1, 0)$ and $T(-1, 1) = (0, 1)$. Find $T(2, 0)$ and $T(0, 3)$.

Linear Transformation and Bases In Exercises 25–28, let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 4, -1)$, $T(0, 1, 0) = (1, 3, -2)$, and $T(0, 0, 1) = (0, -2, 2)$. Find the specified image.

- $T(1, -3, 0)$
- $T(2, -1, 0)$
- $T(2, -4, 1)$
- $T(-2, 4, -1)$

Linear Transformation and Bases In Exercises 29–32, let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(0, -1, 2) = (-3, 2, -1)$, and $T(1, 0, 1) = (1, 1, 0)$. Find the specified image.

- $T(4, 2, 0)$
- $T(0, 2, -1)$
- $T(2, -1, 1)$
- $T(-2, 1, 0)$

Linear Transformation Given by a Matrix In Exercises 33–38, define the linear transformation $T: R^n \rightarrow R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m .

$$33. A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad 34. A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$$

$$35. A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$36. A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$

$$37. A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$38. A = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix}$$

39. For the linear transformation from Exercise 33, find (a) $T(1, 1)$, (b) the preimage of $(1, 1)$, and (c) the preimage of $(0, 0)$.
40. **Writing** For the linear transformation from Exercise 34, find (a) $T(2, 4)$ and (b) the preimage of $(-1, 2, 2)$. (c) Then explain why the vector $(1, 1, 1)$ has no preimage under this transformation.
41. For the linear transformation from Exercise 35, find (a) $T(2, 1, 2, 1)$ and (b) the preimage of $(-1, -1, -1, -1)$.
42. For the linear transformation from Exercise 36, find (a) $T(1, 0, -1, 3, 0)$ and (b) the preimage of $(-1, 8)$.
43. For the linear transformation from Exercise 37, find (a) $T(1, 0, 2, 3)$ and (b) the preimage of $(0, 0, 0)$.
44. For the linear transformation from Exercise 38, find (a) $T(0, 1, 0, 1, 0)$ (b) the preimage of $(0, 0, 0)$, and (c) the preimage of $(1, -1, 2)$.
45. Let T be a linear transformation from R^2 into R^2 such that $T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$. Find (a) $T(4, 4)$ for $\theta = 45^\circ$, (b) $T(4, 4)$ for $\theta = 30^\circ$, and (c) $T(5, 0)$ for $\theta = 120^\circ$.
46. For the linear transformation from Exercise 45, let $\theta = 45^\circ$ and find the preimage of $\mathbf{v} = (1, 1)$.
47. Find the inverse of the matrix A in Example 7. What linear transformation from R^2 into R^2 does A^{-1} represent?
48. For the linear transformation $T: R^2 \rightarrow R^2$ given by

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

find a and b such that $T(12, 5) = (13, 0)$.

Projection in R^3 In Exercises 49 and 50, let the matrix A represent the linear transformation $T: R^3 \rightarrow R^3$. Describe the orthogonal projection to which T maps every vector in R^3 .

$$49. A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 50. A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Transformation Given by a Matrix In Exercises 51–54, determine whether the function involving the $n \times n$ matrix A is a linear transformation.

51. $T: M_{n,n} \rightarrow M_{n,n}, T(A) = A^{-1}$
52. $T: M_{n,n} \rightarrow M_{n,n}, T(A) = AX - XA$, where X is a fixed $n \times m$ matrix
53. $T: M_{n,n} \rightarrow M_{n,m}, T(A) = AB$, where B is a fixed $n \times m$ matrix
54. $T: M_{n,n} \rightarrow R, T(A) = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}$, where $A = [a_{ij}]$
55. Let T be a linear transformation from P_2 into P_2 such that $T(1) = x$, $T(x) = 1 + x$, and $T(x^2) = 1 + x + x^2$. Find $T(2 - 6x + x^2)$.

56. Let T be a linear transformation from $M_{2,2}$ into $M_{2,2}$ such that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\right)$.

Calculus In Exercises 57–60, let D_x be the linear transformation from $C[a, b]$ into $C[a, b]$ from Example 10. Determine whether each statement is true or false. Explain.

57. $D_x(e^{x^2} + 2x) = D_x(e^{x^2}) + 2D_x(x)$

58. $D_x(x^2 - \ln x) = D_x(x^2) - D_x(\ln x)$

59. $D_x(\sin 3x) = 3D_x(\sin x)$

60. $D_x\left(\cos \frac{x}{2}\right) = \frac{1}{2}D_x(\cos x)$

Calculus In Exercises 61–64, for the linear transformation from Example 10, find the preimage of each function.

61. $D_x(f) = 4x + 3$

62. $D_x(f) = e^x$

63. $D_x(f) = \sin x$

64. $D_x(f) = \frac{1}{x}$

65. **Calculus** Let T be a linear transformation from P into R such that

$$T(p) = \int_0^1 p(x) dx.$$

Find (a) $T(-2 + 3x^2)$, (b) $T(x^3 - x^5)$, and (c) $T(-6 + 4x)$.

66. **Calculus** Let T be the linear transformation from P_2 into R using the integral in Exercise 65. Find the preimage of 1. That is, find the polynomial function(s) of degree 2 or less such that $T(p) = 1$.

True or False? In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

67. (a) The function $f(x) = \cos x$ is a linear transformation from R into R .
- (b) For polynomials, the differential operator D_x is a linear transformation from P_n into P_{n-1} .
68. (a) The function $g(x) = x^3$ is a linear transformation from R into R .
- (b) Any linear function of the form $f(x) = ax + b$ is a linear transformation from R into R .

69. Writing Let $T: R^2 \rightarrow R^2$ such that $T(1, 0) = (1, 0)$ and $T(0, 1) = (0, 0)$.

- (a) Determine $T(x, y)$ for (x, y) in R^2 .
- (b) Give a geometric description of T .

70. Writing Let $T: R^2 \rightarrow R^2$ such that $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 0)$.

- (a) Determine $T(x, y)$ for (x, y) in R^2 .
- (b) Give a geometric description of T .

71. Proof Let T be the function that maps R^2 into R^2 such that $T(\mathbf{u}) = \text{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} = (1, 1)$.

- (a) Find $T(x, y)$. (b) Find $T(5, 0)$.
- (c) Prove that T is a linear transformation from R^2 into R^2 .

72. Writing Find $T(3, 4)$ and $T(T(3, 4))$ from Exercise 71 and give geometric descriptions of the results.

73. Show that T from Exercise 71 is represented by the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

74. CAPSTONE Explain how to determine whether a function $T: V \rightarrow W$ is a linear transformation.

75. Proof Use the concept of a fixed point of a linear transformation $T: V \rightarrow V$. A vector $\mathbf{u}$ is a **fixed point** when $T(\mathbf{u}) = \mathbf{u}$.

- (a) Prove that $\mathbf{0}$ is a fixed point of any linear transformation $T: V \rightarrow V$.
- (b) Prove that the set of fixed points of a linear transformation $T: V \rightarrow V$ is a subspace of V .
- (c) Determine all fixed points of the linear transformation $T: R^2 \rightarrow R^2$ represented by $T(x, y) = (x, 2y)$.
- (d) Determine all fixed points of the linear transformation $T: R^2 \rightarrow R^2$ represented by $T(x, y) = (y, x)$.

76. A translation in R^2 is a function of the form $T(x, y) = (x - h, y - k)$, where at least one of the constants h and k is nonzero.

- (a) Show that a translation in R^2 is not a linear transformation.
- (b) For the translation $T(x, y) = (x - 2, y + 1)$, determine the images of $(0, 0)$, $(2, -1)$, and $(5, 4)$.
- (c) Show that a translation in R^2 has no fixed points.

77. Proof Prove that (a) the zero transformation and (b) the identity transformation are linear transformations.

78. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of linearly independent vectors in R^3 . Find a linear transformation T from R^3 into R^3 such that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

79. Proof Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of linearly dependent vectors in V , and let T be a linear transformation from V into V . Prove that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is linearly dependent.

80. Proof Let V be an inner product space. For a fixed vector $\mathbf{v}_0$ in V , define $T: V \rightarrow V$ by $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle \mathbf{v}_0$. Prove that T is a linear transformation.

81. Proof Define $T: M_{n,n} \rightarrow R$ by

$$T(A) = a_{11} + a_{22} + \dots + a_{nn}$$

(the trace of A). Prove that T is a linear transformation.

82. Let V be an inner product space with a subspace W having $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ as an orthonormal basis. Show that the function $T: V \rightarrow W$ represented by

$$T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{v}, \mathbf{w}_2 \rangle \mathbf{w}_2 + \dots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$$

is a linear transformation. T is called the **orthogonal projection of V onto W** .

83. Guided Proof Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for a vector space V . Prove that if a linear transformation $T: V \rightarrow V$ satisfies $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, \dots, n$, then T is the zero transformation.

Getting Started: To prove that T is the zero transformation, you need to show that $T(\mathbf{v}) = \mathbf{0}$ for every vector $\mathbf{v}$ in V .

(i) Let $\mathbf{v}$ be an arbitrary vector in V such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n.$$

(ii) Use the definition and properties of linear transformations to rewrite $T(\mathbf{v})$ as a linear combination of $T(\mathbf{v}_i)$.

(iii) Use the fact that $T(\mathbf{v}_i) = \mathbf{0}$ to conclude that $T(\mathbf{v}) = \mathbf{0}$, making T the zero transformation.

84. Guided Proof Prove that $T: V \rightarrow W$ is a linear transformation if and only if

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars a and b .

Getting Started: This is an “if and only if” statement, so you need to prove the statement in both directions. To prove that T is a linear transformation, you need to show that the function satisfies the definition of a linear transformation. In the other direction, let T be a linear transformation. Use the definition and properties of a linear transformation to prove that $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

(i) Let $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$. Show that T preserves the properties of vector addition and scalar multiplication by choosing appropriate values of a and b .

(ii) To prove the statement in the other direction, assume that T is a linear transformation. Use the properties and definition of a linear transformation to show that $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

6.2 The Kernel and Range of a Linear Transformation

-  Find the kernel of a linear transformation.
-  Find a basis for the range, the rank, and the nullity of a linear transformation.
-  Determine whether a linear transformation is one-to-one or onto.
-  Determine whether vector spaces are isomorphic.

THE KERNEL OF A LINEAR TRANSFORMATION

You know from Theorem 6.1 that for any linear transformation $T: V \rightarrow W$, the zero vector in V maps to the zero vector in W . That is, $T(\mathbf{0}) = \mathbf{0}$. The first question you will consider in this section is whether there are *other* vectors $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$. The collection of all such elements is the **kernel** of T . Note that the symbol $\mathbf{0}$ represents the zero vector in both V and W , although these two zero vectors are often different.

Definition of Kernel of a Linear Transformation

Let $T: V \rightarrow W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is the **kernel** of T and is denoted by $\ker(T)$.

Sometimes the kernel of a transformation can be found by inspection, as demonstrated in Examples 1, 2, and 3.

EXAMPLE 1 Finding the Kernel of a Linear Transformation

Let $T: M_{3,2} \rightarrow M_{2,3}$ be the linear transformation that maps a 3×2 matrix A to its transpose. That is, $T(A) = A^T$. Find the kernel of T .

SOLUTION

For this linear transformation, the 3×2 zero matrix is clearly the only matrix in $M_{3,2}$ whose transpose is the zero matrix in $M_{2,3}$. So, the kernel of T consists of a single element: the zero matrix in $M_{3,2}$.

EXAMPLE 2 The Kernels of the Zero and Identity Transformations

- a. The kernel of the zero transformation $T: V \rightarrow W$ consists of all of V because $T(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v}$ in V . That is, $\ker(T) = V$.
- b. The kernel of the identity transformation $T: V \rightarrow V$ consists of the single element $\mathbf{0}$. That is, $\ker(T) = \{\mathbf{0}\}$.

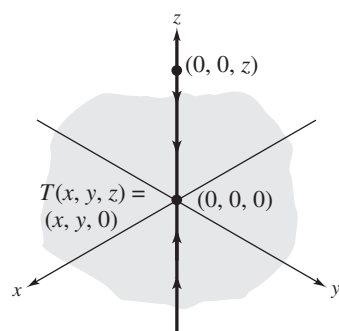
EXAMPLE 3 Finding the Kernel of a Linear Transformation

Find the kernel of the projection $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represented by $T(x, y, z) = (x, y, 0)$.

SOLUTION

This linear transformation projects the vector (x, y, z) in $\mathbb{R}^3$ to the vector $(x, y, 0)$ in the xy -plane. The kernel consists of all vectors lying on the z -axis. That is,

$$\ker(T) = \{(0, 0, z) : z \text{ is a real number}\}. \quad (\text{See Figure 6.1.})$$



The kernel of T is the set of all vectors on the z -axis.

Figure 6.1

Finding the kernels of the linear transformations in Examples 1, 2, and 3 is relatively easy. Sometimes, the kernel of a linear transformation is not so obvious, as illustrated in the next two examples.

EXAMPLE 4 Finding the Kernel of a Linear Transformation

Find the kernel of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ represented by

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1).$$

SOLUTION

To find $\ker(T)$, you need to find all $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ such that

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1) = (0, 0, 0).$$

This leads to the homogeneous system

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 0 &= 0 \\ -x_1 &= 0 \end{aligned}$$

which has only the trivial solution $(x_1, x_2) = (0, 0)$. So, you have

$$\ker(T) = \{(0, 0)\} = \{\mathbf{0}\}.$$

EXAMPLE 5 Finding the Kernel of a Linear Transformation

Find the kernel of the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

SOLUTION

The kernel of T is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (0, 0)$. From this equation, you can write the homogeneous system

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 - x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + 3x_3 &= 0. \end{aligned}$$

Writing the augmented matrix of this system in reduced row-echelon form produces

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 &= x_3 \\ x_2 &= -x_3. \end{aligned}$$

Using the parameter $t = x_3$ produces the family of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

So, the kernel of T is

$$\ker(T) = \{t(1, -1, 1) : t \text{ is a real number}\} = \text{span}\{(1, -1, 1)\}. \quad (\text{See Figure 6.2.})$$

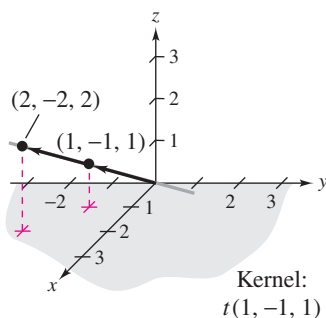


Figure 6.2

Note in Example 5 that the kernel of T contains infinitely many vectors. Of course, the zero vector is in $\ker(T)$, but the kernel also contains such nonzero vectors as $(1, -1, 1)$ and $(2, -2, 2)$, as shown in Figure 6.2. The figure also shows that the kernel is a line passing through the origin, which implies that it is a subspace of $\mathbb{R}^3$. Theorem 6.3 on the next page states that the kernel of every linear transformation $T: V \rightarrow W$ is a subspace of V .

THEOREM 6.3 The Kernel Is a Subspace of V

The kernel of a linear transformation $T: V \rightarrow W$ is a subspace of the domain V .

PROOF

From Theorem 6.1, you know that $\ker(T)$ is a nonempty subset of V . So, by Theorem 4.5, you can show that $\ker(T)$ is a subspace of V by showing that it is closed under vector addition and scalar multiplication. To do so, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T . Then $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, which implies that $\mathbf{u} + \mathbf{v}$ is in the kernel. Moreover, if c is any scalar, then $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$, which implies that $c\mathbf{u}$ is in the kernel. 

REMARK

The kernel of T is sometimes called the **nullspace** of T .

The next example shows how to find a basis for the kernel of a transformation defined by a matrix.

EXAMPLE 6**Finding a Basis for the Kernel**

Define $T: R^5 \rightarrow R^4$ by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for $\ker(T)$ as a subspace of R^5 .

SOLUTION

Using the procedure shown in Example 5, write the augmented matrix $[A \ \mathbf{0}]$ in reduced row-echelon form as shown below.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{l} x_1 = -2x_3 + x_5 \\ x_2 = x_3 + 2x_5 \\ x_4 = -4x_5 \end{array}$$

Letting $x_3 = s$ and $x_5 = t$, you have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s \\ 0s - 4t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

So a basis for the kernel of T is $B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$. 

DISCOVERY

1. What is the rank of the matrix A in Example 6?
2. Formulate a conjecture relating the dimension of the kernel, the rank, and the number of columns of A .
3. Verify your conjecture for the matrix in Example 5.

In the solution of Example 6, a basis for the kernel of T was found by solving the homogeneous system represented by $A\mathbf{x} = \mathbf{0}$. This procedure is a familiar one—it is the same procedure used to find the *nullspace* of A . In other words, the kernel of T is the solution space of $A\mathbf{x} = \mathbf{0}$, as stated in the corollary to Theorem 6.3 below.

THEOREM 6.3 Corollary

Let $T: R^n \rightarrow R^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of T is equal to the solution space of $A\mathbf{x} = \mathbf{0}$.

THE RANGE OF A LINEAR TRANSFORMATION

The kernel is one of two critical subspaces associated with a linear transformation. The other is the range of T , denoted by $\text{range}(T)$. Recall from Section 6.1 that the range of $T: V \rightarrow W$ is the set of all vectors $\mathbf{w}$ in W that are images of vectors in V . That is,

$$\text{range}(T) = \{T(\mathbf{v}) : \mathbf{v} \text{ is in } V\}.$$

THEOREM 6.4 The Range of T Is a Subspace of W

The range of a linear transformation $T: V \rightarrow W$ is a subspace of W .

PROOF

The range of T is nonempty because $T(\mathbf{0}) = \mathbf{0}$ implies that the range contains the zero vector. To show that it is closed under vector addition, let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vectors in the range of T . The vectors $\mathbf{u}$ and $\mathbf{v}$ are in V , so it follows that $\mathbf{u} + \mathbf{v}$ is also in V , and the sum

$$T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$$

is in the range of T .

To show closure under scalar multiplication, let $T(\mathbf{u})$ be a vector in the range of T and let c be a scalar. $\mathbf{u}$ is in V , so it follows that $c\mathbf{u}$ is also in V , and the scalar multiple $cT(\mathbf{u}) = T(c\mathbf{u})$ is in the range of T . 

Note that the kernel and range of a linear transformation $T: V \rightarrow W$ are subspaces of V and W , respectively, as illustrated in Figure 6.3.

To find a basis for the range of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$, observe that the range consists of all vectors $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ is consistent. Writing the system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in the form

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}$$

shows that $\mathbf{b}$ is in the range of T if and only if $\mathbf{b}$ is a linear combination of the column vectors of A . So *the column space of the matrix A is the same as the range of T .*

THEOREM 6.4 Corollary

Let $T: R^n \rightarrow R^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Then the column space of A is equal to the range of T .

In Examples 4 and 5 in Section 4.6, you saw two procedures for finding a basis for the column space of a matrix. The next example uses the procedure from Example 5 in Section 4.6 to find a basis for the range of a linear transformation defined by a matrix.

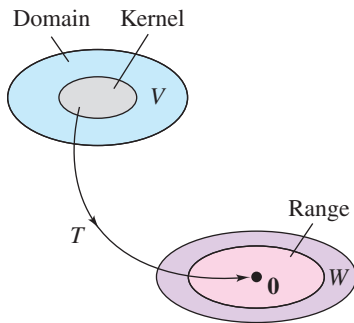


Figure 6.3

EXAMPLE 7**Finding a Basis for the Range of a Linear Transformation**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

For the linear transformation $R^5 \rightarrow R^4$ from Example 6, find a basis for the range of T .

SOLUTION

Use the reduced row-echelon form of A from Example 6.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's appear in columns 1, 2, and 4 of the reduced matrix on the right, so the corresponding column vectors of A form a basis for the column space of A . A basis for the range of T is $B = \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$. 

REMARK

If T is given by a matrix A , then the rank of T is equal to the rank of A , and the nullity of T is equal to the nullity of A , as defined in Section 4.6.

The next definition gives the dimensions of the kernel and range of a linear transformation.

Definition of Rank and Nullity of a Linear Transformation

Let $T: V \rightarrow W$ be a linear transformation. The dimension of the kernel of T is called the **nullity** of T and is denoted by **nullity**(T). The dimension of the range of T is called the **rank** of T and is denoted by **rank**(T).

In Examples 6 and 7, the rank and nullity of T are related to the dimension of the domain as shown below.

$$\text{rank}(T) + \text{nullity}(T) = 3 + 2 = 5 = \text{dimension of domain}$$

This relationship is true for any linear transformation from a finite-dimensional vector space, as stated in the next theorem.

THEOREM 6.5 Sum of Rank and Nullity

Let $T: V \rightarrow W$ be a linear transformation from an n -dimensional vector space V into a vector space W . Then the sum of the dimensions of the range and kernel is equal to the dimension of the domain. That is,

$$\text{rank}(T) + \text{nullity}(T) = n \quad \text{or} \quad \dim(\text{range}) + \dim(\text{kernel}) = \dim(\text{domain}).$$

PROOF

The proof provided here covers the case in which T is represented by an $m \times n$ matrix A . The general case will follow in the next section, where you will see that any linear transformation from an n -dimensional space into an m -dimensional space can be represented by a matrix. To prove this theorem, assume that the matrix A has a rank of r . Then you have

$$\text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space}) = \text{rank}(A) = r.$$

From Theorem 4.17, however, you know that

$$\text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } \mathbf{Ax} = \mathbf{0}) = n - r.$$

So, it follows that $\text{rank}(T) + \text{nullity}(T) = r + (n - r) = n$. 

EXAMPLE 8**Finding the Rank and Nullity of a Linear Transformation**

Find the rank and nullity of the linear transformation $T: R^3 \rightarrow R^3$ defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

SOLUTION

A is in reduced row-echelon form and has two nonzero rows, so it has a rank of 2. This means that the rank of T is also 2, and the nullity is $\dim(\text{domain}) - \text{rank} = 3 - 2 = 1$.

One way to visualize the relationship between the rank and the nullity of a linear transformation provided by a matrix in row-echelon form is to observe that the number of leading 1's determines the rank, and the number of free variables (columns without leading 1's) determines the nullity. Their sum must be the total number of columns in the matrix, which is the dimension of the domain. In Example 8, the first two columns have leading 1's, indicating that the rank is 2. The third column corresponds to a free variable, indicating that the nullity is 1.

EXAMPLE 9**Finding the Rank and Nullity of a Linear Transformation**

Let $T: R^5 \rightarrow R^7$ be a linear transformation.

- Find the dimension of the kernel of T when the dimension of the range is 2.
- Find the rank of T when the nullity of T is 4.
- Find the rank of T when $\ker(T) = \{\mathbf{0}\}$.

SOLUTION

- By Theorem 6.5, with $n = 5$, you have

$$\dim(\text{kernel}) = n - \dim(\text{range}) = 5 - 2 = 3.$$

- Again by Theorem 6.5, you have

$$\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1.$$

- In this case, the nullity of T is 0. So

$$\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5.$$

**LINEAR ALGEBRA APPLIED**

A control system, such as the one shown for a dairy factory, processes an input signal $\mathbf{x}_k$ and produces an output signal $\mathbf{x}_{k+1}$. Without external feedback, the **difference equation** $\mathbf{x}_{k+1} = A\mathbf{x}_k$, a linear transformation where $\mathbf{x}_i$ is an $n \times 1$ vector and A is an $n \times n$ matrix, can model the relationship between the input and output signals. Typically, however, a control system has external feedback, so the relationship becomes $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, where B is an $n \times m$ matrix and $\mathbf{u}_k$ is an $m \times 1$ input, or control, vector. A system is *controllable* when it can reach any desired final state from its initial state in n or fewer steps. If A and B are matrices in a model of a controllable system, then the rank of the *controllability matrix*

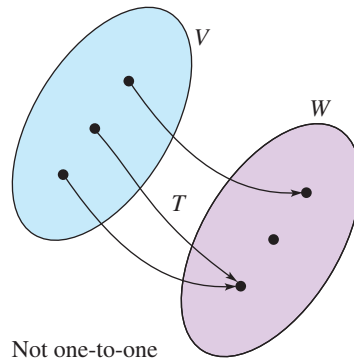
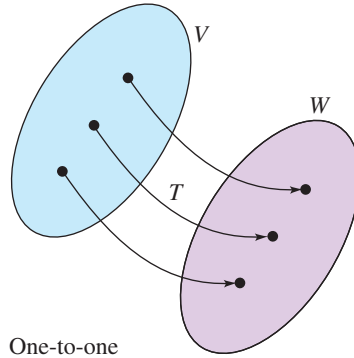
$$[B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

is equal to n .

Mark Yuill/Shutterstock.com

ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

This section began with a question: Which vectors in the domain of a linear transformation are mapped to the zero vector? Theorem 6.6 (below) states that if the zero vector is the only vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$, then T is *one-to-one*. A function $T: V \rightarrow W$ is **one-to-one** when the preimage of every $\mathbf{w}$ in the range consists of a single vector, as shown below. This is equivalent to saying that T is one-to-one if and only if, for all $\mathbf{u}$ and $\mathbf{v}$ in V , $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.



THEOREM 6.6 One-to-One Linear Transformations

Let $T: V \rightarrow W$ be a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

PROOF

First assume that T is one-to-one. Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution: $\mathbf{v} = \mathbf{0}$. In that case, $\ker(T) = \{\mathbf{0}\}$. Conversely, assume that $\ker(T) = \{\mathbf{0}\}$ and $T(\mathbf{u}) = T(\mathbf{v})$. You know that T is a linear transformation, so it follows that

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}.$$

This implies that the vector $\mathbf{u} - \mathbf{v}$ lies in the kernel of T and must equal $\mathbf{0}$. So, $\mathbf{u} = \mathbf{v}$, which means that T is one-to-one. 

EXAMPLE 10

One-to-One and Not One-to-One Linear Transformations

- a. The linear transformation $T: M_{m,n} \rightarrow M_{n,m}$ represented by $T(A) = A^T$ is one-to-one because its kernel consists of only the $m \times n$ zero matrix.
- b. The zero transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is not one-to-one because its kernel is all of $\mathbb{R}^3$. 

A function $T: V \rightarrow W$ is **onto** when every element in W has a preimage in V . In other words, T is onto when W is equal to the range of T . The proof of the related theorem below is left as an exercise. (See Exercise 69.)

THEOREM 6.7 **Onto Linear Transformations**

Let $T: V \rightarrow W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the rank of T is equal to the dimension of W .

For vector spaces of equal dimensions, you can combine the results of Theorems 6.5, 6.6, and 6.7 to obtain the next theorem relating the concepts of one-to-one and onto.

THEOREM 6.8 **One-to-One and Onto Linear Transformations**

Let $T: V \rightarrow W$ be a linear transformation with vector spaces V and W , both of dimension n . Then T is one-to-one if and only if it is onto.

PROOF

If T is one-to-one, then by Theorem 6.6 $\ker(T) = \{0\}$, and $\dim(\ker(T)) = 0$. In that case, Theorem 6.5 produces

$$\dim(\text{range of } T) = n - \dim(\ker(T)) = n = \dim(W).$$

Consequently, by Theorem 6.7, T is onto. Similarly, if T is onto, then

$$\dim(\text{range of } T) = \dim(W) = n$$

which by Theorem 6.5 implies that $\dim(\ker(T)) = 0$. By Theorem 6.6, T is one-to-one.

The next example brings together several concepts related to the kernel and range of a linear transformation.

EXAMPLE 11 **Summarizing Several Results**

Consider the linear transformation $T: R^n \rightarrow R^m$ represented by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T , and determine whether T is one-to-one, onto, or neither.

a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

SOLUTION

Note that each matrix is already in row-echelon form, so its rank can be determined by inspection.

	$T: R^n \rightarrow R^m$	Dim(domain)	Dim(range) Rank(T)	Dim(kernel) Nullity(T)	One-to-One	Onto
a.	$T: R^3 \rightarrow R^3$	3	3	0	Yes	Yes
b.	$T: R^2 \rightarrow R^3$	2	2	0	Yes	No
c.	$T: R^3 \rightarrow R^2$	3	2	1	No	Yes
d.	$T: R^3 \rightarrow R^3$	3	2	1	No	No

ISOMORPHISMS OF VECTOR SPACES

Distinct vector spaces such as R^3 and $M_{3,1}$ can be thought of as being “essentially the same”—at least with respect to the operations of vector addition and scalar multiplication. Such spaces are **isomorphic** to each other. (The Greek word *isos* means “equal.”)

Definition of Isomorphism

A linear transformation $T: V \rightarrow W$ that is one-to-one and onto is called an **isomorphism**. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W , then V and W are **isomorphic** to each other.

Isomorphic vector spaces are of the same finite dimension, and vector spaces of the same finite dimension are isomorphic, as stated in the next theorem.

THEOREM 6.9 Isomorphic Spaces and Dimension

Two finite-dimensional vector spaces V and W are isomorphic if and only if they are of the same dimension.

PROOF

Assume V is isomorphic to W , where V has dimension n . By the definition of isomorphic spaces, you know there exists a linear transformation $T: V \rightarrow W$ that is one-to-one and onto. T is one-to-one, so it follows that $\dim(\text{kernel}) = 0$, which also implies that

$$\dim(\text{range}) = \dim(\text{domain}) = n.$$

In addition, T is onto, so you can conclude that $\dim(\text{range}) = \dim(W) = n$.

To prove the theorem in the other direction, assume V and W both have dimension n . Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V , and let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be a basis for W . Then an arbitrary vector in V can be represented as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

and you can define a linear transformation $T: V \rightarrow W$ as shown below.

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \dots + c_n\mathbf{w}_n$$

Verify that this linear transformation is both one-to-one and onto. So, V and W are isomorphic. 

Example 12 lists some vector spaces that are isomorphic to R^4 .

REMARK

Your study of vector spaces has included much greater coverage to R^n than to other vector spaces. This preference for R^n stems from its notational convenience and from the geometric models available for R^2 and R^3 .

EXAMPLE 12

Isomorphic Vector Spaces

The vector spaces below are isomorphic to each other.

- $R^4 = 4\text{-space}$
 - $M_{4,1} = \text{space of all } 4 \times 1 \text{ matrices}$
 - $M_{2,2} = \text{space of all } 2 \times 2 \text{ matrices}$
 - $P_3 = \text{space of all polynomials of degree 3 or less}$
 - $V = \{(x_1, x_2, x_3, x_4, 0) : x_i \text{ is a real number}\}$ (subspace of R^5)
- 

Example 12 tells you that the elements in these spaces behave in the same way as an arbitrary vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$.

6.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Kernel of a Linear Transformation****In Exercises 1–10, find the kernel of the linear transformation.**

1. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (0, 0, 0)$
2. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x, 0, z)$
3. $T: R^4 \rightarrow R^4$, $T(x, y, z, w) = (y, x, w, z)$
4. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (-z, -y, -x)$
5. $T: P_3 \rightarrow R$,
 $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + a_2$
6. $T: P_2 \rightarrow R$, $T(a_0 + a_1x + a_2x^2) = a_0$
7. $T: P_2 \rightarrow P_1$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$
8. $T: P_3 \rightarrow P_2$,
 $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$
9. $T: R^2 \rightarrow R^2$, $T(x, y) = (x + 2y, y - x)$
10. $T: R^2 \rightarrow R^2$, $T(x, y) = (x - y, y - x)$

Finding the Kernel and Range In Exercises 11–18, define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Find (a) the kernel of T and (b) the range of T .

11. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
12. $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$
13. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$
14. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$
15. $A = \begin{bmatrix} 1 & 3 \\ -1 & -3 \\ 2 & 2 \end{bmatrix}$
16. $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$
17. $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$
18. $A = \begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix}$

Finding the Kernel, Nullity, Range, and Rank In Exercises 19–32, define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Find (a) $\ker(T)$, (b) $\text{nullity}(T)$, (c) $\text{range}(T)$, and (d) $\text{rank}(T)$.

19. $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$
20. $A = \begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$
21. $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$
22. $A = \begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 2 & -3 \end{bmatrix}$
23. $A = \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$
24. $A = \begin{bmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
26. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
27. $A = \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}$
28. $A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$
29. $A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$
30. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
31. $A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$
32. $A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix}$

Finding the Nullity and Describing the Kernel and Range In Exercises 33–40, let $T: R^3 \rightarrow R^3$ be a linear transformation. Find the nullity of T and give a geometric description of the kernel and range of T .

33. $\text{rank}(T) = 2$
34. $\text{rank}(T) = 1$
35. $\text{rank}(T) = 0$
36. $\text{rank}(T) = 3$
37. T is the counterclockwise rotation of 45° about the z -axis:
$$T(x, y, z) = \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y, z \right)$$
38. T is the reflection through the yz -coordinate plane:
 $T(x, y, z) = (-x, y, z)$
39. T is the projection onto the vector $\mathbf{v} = (1, 2, 2)$:
$$T(x, y, z) = \frac{x + 2y + 2z}{9}(1, 2, 2)$$
40. T is the projection onto the xy -coordinate plane:
 $T(x, y, z) = (x, y, 0)$

Finding the Nullity of a Linear Transformation In Exercises 41–46, find the nullity of T .

41. $T: R^4 \rightarrow R^2$, $\text{rank}(T) = 2$
42. $T: R^4 \rightarrow R^4$, $\text{rank}(T) = 0$
43. $T: P_5 \rightarrow P_2$, $\text{rank}(T) = 3$
44. $T: P_3 \rightarrow P_1$, $\text{rank}(T) = 2$
45. $T: M_{2,4} \rightarrow M_{4,2}$, $\text{rank}(T) = 4$
46. $T: M_{3,3} \rightarrow M_{2,3}$, $\text{rank}(T) = 6$

Verifying That T Is One-to-One and Onto In Exercises 47–50, verify that the matrix defines a linear function T that is one-to-one and onto.

47. $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

49. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

50. $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

Determining Whether T Is One-to-One, Onto, or Neither In Exercises 51–54, determine whether the linear transformation is one-to-one, onto, or neither.

51. T in Exercise 3

52. T in Exercise 10

53. $T: R^2 \rightarrow R^3$, $T(\mathbf{x}) = A\mathbf{x}$, where A is given in Exercise 21

54. $T: R^5 \rightarrow R^3$, $T(\mathbf{x}) = A\mathbf{x}$, where A is given in Exercise 18

55. Identify the zero element and standard basis for each of the isomorphic vector spaces in Example 12.

56. Which vector spaces are isomorphic to R^6 ?

- (a) $M_{2,3}$ (b) P_6 (c) $C[0, 6]$
 (d) $M_{6,1}$ (e) P_5 (f) $C'[-3, 3]$
 (g) $\{(x_1, x_2, x_3, 0, x_5, x_6, x_7): x_i \text{ is a real number}\}$

57. **Calculus** Define $T: P_4 \rightarrow P_3$ by $T(p) = p'$. What is the kernel of T ?

58. **Calculus** Define $T: P_2 \rightarrow R$ by

$$T(p) = \int_0^1 p(x) dx.$$

What is the kernel of T ?

59. Let $T: R^3 \rightarrow R^3$ be the linear transformation that projects $\mathbf{u}$ onto $\mathbf{v} = (2, -1, 1)$.

- (a) Find the rank and nullity of T .
 (b) Find a basis for the kernel of T .

60. CAPSTONE Let $T: R^4 \rightarrow R^3$ be the linear transformation represented by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- (a) Find the dimension of the domain.
 (b) Find the dimension of the range.
 (c) Find the dimension of the kernel.
 (d) Is T one-to-one? Explain.
 (e) Is T onto? Explain.
 (f) Is T an isomorphism? Explain.

61. For the transformation $T: R^n \rightarrow R^n$ represented by $T(\mathbf{x}) = A\mathbf{x}$, what can be said about the rank of T when (a) $\det(A) \neq 0$ and (b) $\det(A) = 0$?

62. Writing Let $T: R^m \rightarrow R^n$ be a linear transformation. Explain the differences between the concepts of one-to-one and onto. What can you say about m and n when T is onto? What can you say about m and n when T is one-to-one?

63. Define $T: M_{n,n} \rightarrow M_{n,n}$ by $T(A) = A - A^T$. Show that the kernel of T is the set of $n \times n$ symmetric matrices.

64. Determine a relationship among m , n , j , and k such that $M_{m,n}$ is isomorphic to $M_{j,k}$.

True or False? In Exercises 65 and 66, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

65. (a) The set of all vectors mapped from a vector space V into another vector space W by a linear transformation T is the kernel of T .

(b) The range of a linear transformation from a vector space V into a vector space W is a subspace of V .

(c) The vector spaces R^3 and $M_{3,1}$ are isomorphic to each other.

66. (a) The dimension of a linear transformation T from a vector space V into a vector space W is the rank of T .

(b) A linear transformation T from V into W is one-to-one when the preimage of every $\mathbf{w}$ in the range consists of a single vector $\mathbf{v}$.

(c) The vector spaces R^2 and P_1 are isomorphic to each other.

67. Guided Proof Let B be an invertible $n \times n$ matrix. Prove that the linear transformation $T: M_{n,n} \rightarrow M_{n,n}$ represented by $T(A) = AB$ is an isomorphism.

Getting Started: To show that the linear transformation is an isomorphism, you need to show that T is both onto and one-to-one.

(i) T is a linear transformation with vector spaces of equal dimension, so by Theorem 6.8, you only need to show that T is one-to-one.

(ii) To show that T is one-to-one, you need to determine the kernel of T and show that it is $\{\mathbf{0}\}$ (Theorem 6.6). Use the fact that B is an invertible $n \times n$ matrix and that $T(A) = AB$.

(iii) Conclude that T is an isomorphism.

68. Proof Let $T: V \rightarrow W$ be a linear transformation. Prove that T is one-to-one if and only if the rank of T equals the dimension of V .

69. Proof Prove Theorem 6.7.

70. Proof Let $T: V \rightarrow W$ be a linear transformation, and let U be a subspace of W . Prove that the set $T^{-1}(U) = \{\mathbf{v} \in V: T(\mathbf{v}) \in U\}$ is a subspace of V . What is $T^{-1}(U)$ when $U = \{\mathbf{0}\}$?

6.3 Matrices for Linear Transformations

-  Find the standard matrix for a linear transformation.
-  Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation.
-  Find the matrix for a linear transformation relative to a nonstandard basis.

THE STANDARD MATRIX FOR A LINEAR TRANSFORMATION

Which representation of $T: R^3 \rightarrow R^3$ is better:

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

or

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}?$$

The second representation is better than the first for at least three reasons: it is simpler to write, simpler to read, and easier to enter into a calculator or math software. Later, you will see that matrix representation of linear transformations also has some theoretical advantages. In this section, you will see that for linear transformations involving finite-dimensional vector spaces, matrix representation is always possible.

The key to representing a linear transformation $T: V \rightarrow W$ by a matrix is to determine how it acts on a basis for V . Once you know the image of every vector in the basis, you can use the properties of linear transformations to determine $T(\mathbf{v})$ for any $\mathbf{v}$ in V .

Recall that the standard basis for R^n , written in column vector notation, is

$$\begin{aligned} B &= \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} \\ &= \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

THEOREM 6.10 Standard Matrix for a Linear Transformation

Let $T: R^n \rightarrow R^m$ be a linear transformation such that, for the standard basis vectors $\mathbf{e}_i$ of R^n ,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n . A is called the **standard matrix** for T .

PROOF

To show that $T(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v}$ in R^n , you can write

$$\mathbf{v} = [v_1 \ v_2 \ \dots \ v_n]^T = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n.$$

T is a linear transformation, so you have

$$\begin{aligned} T(\mathbf{v}) &= T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n) \\ &= T(v_1\mathbf{e}_1) + T(v_2\mathbf{e}_2) + \dots + T(v_n\mathbf{e}_n) \\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \dots + v_nT(\mathbf{e}_n). \end{aligned}$$

On the other hand, the matrix product $A\mathbf{v}$ is

$$\begin{aligned} A\mathbf{v} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{bmatrix} \\ &= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \dots + v_nT(\mathbf{e}_n). \end{aligned}$$

So, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n .

EXAMPLE 1**Finding the Standard Matrix for a Linear Transformation**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the standard matrix for the linear transformation $T: R^3 \rightarrow R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y).$$

SOLUTION

Begin by finding the images of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

Vector Notation

$$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 2)$$

$$T(\mathbf{e}_2) = T(0, 1, 0) = (-2, 1)$$

$$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0)$$

Matrix Notation

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By Theorem 6.10, the columns of A consist of $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$, and you have

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

REMARK

As a check, note that

$$\begin{aligned} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix} \end{aligned}$$

which is equivalent to

$$T(x, y, z) = (x - 2y, 2x + y).$$

A little practice will enable you to determine the standard matrix for a linear transformation, such as the one in Example 1, by inspection. For example, to find the standard matrix for the linear transformation

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 5x_3, 2x_1 + 3x_3, 4x_1 + x_2 - 2x_3)$$

use the coefficients of x_1 , x_2 , and x_3 to form the rows of A , as shown below.

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & -2 \end{bmatrix} \begin{array}{l} \leftarrow 1x_1 - 2x_2 + 5x_3 \\ \leftarrow 2x_1 + 0x_2 + 3x_3 \\ \leftarrow 4x_1 + 1x_2 - 2x_3 \end{array}$$

EXAMPLE 2

Finding the Standard Matrix for a Linear Transformation

The linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ projects each point in $\mathbb{R}^2$ onto the x -axis, as shown at the right. Find the standard matrix for T .

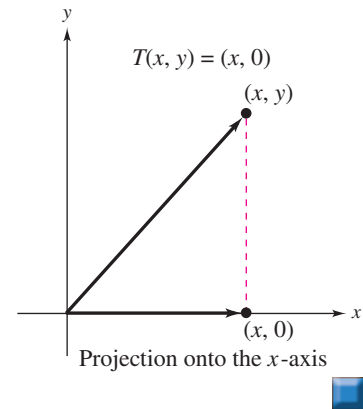
SOLUTION

This linear transformation is represented by

$$T(x, y) = (x, 0).$$

So, the standard matrix for T is

$$\begin{aligned} A &= [T(1, 0) \ T(0, 1)] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$



The standard matrix for the zero transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is the $m \times n$ zero matrix, and the standard matrix for the identity transformation from $\mathbb{R}^n$ into $\mathbb{R}^n$ is I_n .



LINEAR ALGEBRA APPLIED

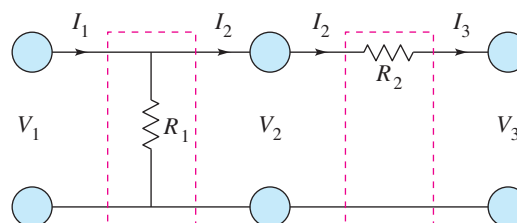
Ladder networks are useful tools for electrical engineers involved in circuit design. In a ladder network, the output voltage V and current I of one circuit are the input voltage and current of the circuit next to it. In the ladder network shown below, linear transformations can relate the input and output of an individual circuit (enclosed in a dashed box). Using Kirchhoff's Voltage and Current Laws and Ohm's Law,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

and

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

A *composition* can relate the input and output of the entire ladder network, that is, V_1 and I_1 to V_3 and I_3 . Discussion on the composition of linear transformations begins on the next page.



any_keen/Shutterstock.com

COMPOSITION OF LINEAR TRANSFORMATIONS

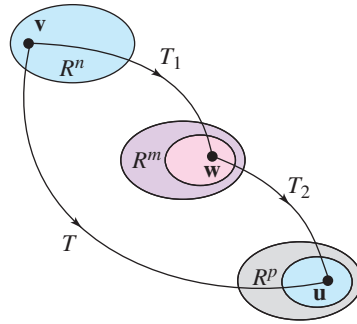
The **composition**, T , of $T_1: R^n \rightarrow R^m$ with $T_2: R^m \rightarrow R^p$ is

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$$

where $\mathbf{v}$ is a vector in R^n . This composition is denoted by

$$T = T_2 \circ T_1.$$

The domain of T is the domain of T_1 . Moreover, the composition is not defined unless the range of T_1 lies within the domain of T_2 , as shown below.



The next theorem emphasizes the usefulness of matrices for representing linear transformations. This theorem not only states that the composition of two linear transformations is a linear transformation, but also says that the standard matrix for the composition is the product of the standard matrices for the two original linear transformations.

THEOREM 6.11 Composition of Linear Transformations

Let $T_1: R^n \rightarrow R^m$ and $T_2: R^m \rightarrow R^p$ be linear transformations with standard matrices A_1 and A_2 , respectively. The **composition** $T: R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is the matrix product

$$A = A_2 A_1.$$

PROOF

To show that T is a linear transformation, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar. T_1 and T_2 are linear transformations, so you can write

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T_2(T_1(\mathbf{u} + \mathbf{v})) \\ &= T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \\ T(c\mathbf{v}) &= T_2(T_1(c\mathbf{v})) \\ &= T_2(cT_1(\mathbf{v})) \\ &= cT_2(T_1(\mathbf{v})) \\ &= cT(\mathbf{v}). \end{aligned}$$

Now, to show that $A_2 A_1$ is the standard matrix for T , use the associative property of matrix multiplication to write

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1 \mathbf{v}) = A_2(A_1 \mathbf{v}) = (A_2 A_1) \mathbf{v}.$$



Theorem 6.11 can be generalized to cover the composition of n linear transformations. That is, if the standard matrices of $T_1, T_2, \dots, T_n$ are $A_1, A_2, \dots, A_n$, respectively, then the standard matrix for the composition $T(\mathbf{v}) = T_n(T_{n-1}(\dots(T_2(T_1(\mathbf{v})))) \dots)$ is represented by $A = A_n A_{n-1} \dots A_2 A_1$.

Matrix multiplication is not commutative, so order is important when forming the compositions of linear transformations. In general, the composition $T_2 \circ T_1$ is not the same as $T_1 \circ T_2$, as demonstrated in the next example.

EXAMPLE 3 The Standard Matrix for a Composition

Let T_1 and T_2 be linear transformations from R^3 into R^3 such that

$$T_1(x, y, z) = (2x + y, 0, x + z) \quad \text{and} \quad T_2(x, y, z) = (x - y, z, y).$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

SOLUTION

The standard matrices for T_1 and T_2 are

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By Theorem 6.11, the standard matrix for T is

$$\begin{aligned} A &= A_2 A_1 \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and the standard matrix for T' is

$$\begin{aligned} A' &= A_1 A_2 \\ &= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Another benefit of matrix representation is that it can represent the **inverse** of a linear transformation. Before seeing how this works, consider the next definition.

Definition of Inverse Linear Transformation

If $T_1: R^n \rightarrow R^n$ and $T_2: R^n \rightarrow R^n$ are linear transformations such that for every $\mathbf{v}$ in R^n ,

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

then T_2 is the **inverse** of T_1 , and T_1 is said to be **invertible**.

Not every linear transformation has an inverse. If the transformation T_1 is invertible, however, then the inverse is unique and is denoted by T_1^{-1} .

Just as the inverse of a function of a real variable can be thought of as undoing what the function did, the inverse of a linear transformation T can be thought of as undoing the mapping done by T . For example, if T is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that

$$T(1, 4, -5) = (2, 3, 1)$$

and if T^{-1} exists, then T^{-1} maps $(2, 3, 1)$ back to its preimage under T . That is,

$$T^{-1}(2, 3, 1) = (1, 4, -5).$$

The next theorem states that a linear transformation is invertible if and only if it is an isomorphism (one-to-one and onto). You are asked to prove this theorem in Exercise 56.

REMARK

Several other conditions are equivalent to the three listed in Theorem 6.12; see the summary of equivalent conditions for square matrices in Section 4.6.

THEOREM 6.12 Existence of an Inverse Transformation

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation with standard matrix A . Then the conditions listed below are equivalent.

1. T is invertible.
2. T is an isomorphism.
3. A is invertible.

If T is invertible with standard matrix A , then the standard matrix for T^{-1} is A^{-1} .

EXAMPLE 4

Finding the Inverse of a Linear Transformation

Consider the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3).$$

Show that T is invertible, and find its inverse.

SOLUTION

The standard matrix for T is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Using the method presented in Section 2.3 or a graphing calculator, you can find that A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

So, T is invertible and the standard matrix for T^{-1} is A^{-1} . 

Using the standard matrix for the inverse, you can find the rule for T^{-1} by computing the image of an arbitrary vector $\mathbf{x} = (x_1, x_2, x_3)$.

$$\begin{aligned} A^{-1}\mathbf{x} &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix} \end{aligned}$$

Or,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3).$$

NONSTANDARD BASES AND GENERAL VECTOR SPACES

You will now consider the more general problem of finding a matrix for a linear transformation $T: V \rightarrow W$, where B and B' are ordered bases for V and W , respectively. Recall that the coordinate matrix of $\mathbf{v}$ relative to B is denoted by $[\mathbf{v}]_B$. To represent the linear transformation T , multiply A by a *coordinate matrix relative to B* to obtain a *coordinate matrix relative to B'* . That is, $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$. The matrix A is called the **matrix of T relative to the bases B and B'** .

To find the matrix A , you will use a procedure similar to the one used to find the standard matrix for T . That is, the images of the vectors in B are written as coordinate matrices relative to the basis B' . These coordinate matrices form the columns of A .

Transformation Matrix for Nonstandard Bases

Let V and W be finite-dimensional vector spaces with bases B and B' , respectively, where

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}.$$

If $T: V \rightarrow W$ is a linear transformation such that

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is such that $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V .

EXAMPLE 5

Finding a Matrix Relative to Nonstandard Bases

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$. Find the matrix for T relative to the bases

$$B = \{\overset{\mathbf{v}_1}{(1, 2)}, \overset{\mathbf{v}_2}{(-1, 1)}\} \quad \text{and} \quad B' = \{\overset{\mathbf{w}_1}{(1, 0)}, \overset{\mathbf{w}_2}{(0, 1)}\}.$$

SOLUTION

By the definition of T , you have

$$T(\mathbf{v}_1) = T(1, 2) = (3, 0) = 3\mathbf{w}_1 + 0\mathbf{w}_2$$

$$T(\mathbf{v}_2) = T(-1, 1) = (0, -3) = 0\mathbf{w}_1 - 3\mathbf{w}_2.$$

The coordinate matrices for $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ relative to B' are

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.$$

Form the matrix for T relative to B and B' by using these coordinate matrices as columns to produce

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}.$$



EXAMPLE 6**Using a Matrix to Represent a Linear Transformation**

For the linear transformation $T: R^2 \rightarrow R^2$ in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$.

SOLUTION

Using the basis $B = \{(1, 2), (-1, 1)\}$, you find that $\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1)$, which implies

$$[\mathbf{v}]_B = [1 \quad -1]^T.$$

So, $[T(\mathbf{v})]_{B'}$ is

$$A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Finally, $B' = \{(1, 0), (0, 1)\}$, so it follows that

$$T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3).$$

Check this result by directly calculating $T(\mathbf{v})$ using the definition of T in Example 5: $T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$. 

For the special case where $V = W$ and $B = B'$, the matrix A is called the **matrix of T relative to the basis B** . In this case, the matrix of the identity transformation is simply I_n . To see this, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. The identity transformation maps each $\mathbf{v}_i$ to itself, so you have $[T(\mathbf{v}_1)]_B = [1 \ 0 \ \dots \ 0]^T$, $[T(\mathbf{v}_2)]_B = [0 \ 1 \ \dots \ 0]^T$, $\dots$, $[T(\mathbf{v}_n)]_B = [0 \ 0 \ \dots \ 1]^T$, and it follows that $A = I_n$.

In the next example, you will construct a matrix representing the differential operator discussed in Example 10 in Section 6.1.

EXAMPLE 7**A Matrix for the Differential Operator (Calculus)**

Let $D_x: P_2 \rightarrow P_1$ be the differential operator that maps a polynomial p of degree 2 or less onto its derivative p' . Find the matrix for D_x using the bases

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1, x\}.$$

SOLUTION

The derivatives of the basis vectors are

$$D_x(1) = 0 = 0(1) + 0(x)$$

$$D_x(x) = 1 = 1(1) + 0(x)$$

$$D_x(x^2) = 2x = 0(1) + 2(x).$$

So, the coordinate matrices relative to B' are

$$[D_x(1)]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [D_x(x)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [D_x(x^2)]_{B'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and the matrix for D_x is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that this matrix *does* produce the derivative of a quadratic polynomial $p(x) = a + bx + cx^2$.

$$Ap = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} \Rightarrow b + 2cx = D_x[a + bx + cx^2]$$


6.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



The Standard Matrix for a Linear Transformation In Exercises 1–6, find the standard matrix for the linear transformation T .

- $T(x, y) = (x + 2y, x - 2y)$
- $T(x, y) = (2x - 3y, x - y, y - 4x)$
- $T(x, y, z) = (x + y, x - y, z - x)$
- $T(x, y) = (5x + y, 0, 4x - 5y)$
- $T(x, y, z) = (3x - 2z, 2y - z)$
- $T(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

Finding the Image of a Vector In Exercises 7–10, use the standard matrix for the linear transformation T to find the image of the vector $\mathbf{v}$.

- $T(x, y, z) = (2x + y, 3y - z), \mathbf{v} = (0, 1, -1)$
- $T(x, y) = (x + y, x - y, 2x, 2y), \mathbf{v} = (3, -3)$
- $T(x, y) = (x - 3y, 2x + y, y), \mathbf{v} = (-2, 4)$
- $T(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_4, x_3 - x_1, x_2 + x_4), \mathbf{v} = (1, 2, 3, -2)$

Finding the Standard Matrix and the Image In Exercises 11–22, (a) find the standard matrix A for the linear transformation T , (b) use A to find the image of the vector $\mathbf{v}$, and (c) sketch the graph of $\mathbf{v}$ and its image.

- T is the reflection in the origin in R^2 : $T(x, y) = (-x, -y), \mathbf{v} = (3, 4)$.
- T is the reflection in the line $y = x$ in R^2 : $T(x, y) = (y, x), \mathbf{v} = (3, 4)$.
- T is the reflection in the y -axis in R^2 : $T(x, y) = (-x, y), \mathbf{v} = (2, -3)$.
- T is the reflection in the x -axis in R^2 : $T(x, y) = (x, -y), \mathbf{v} = (4, -1)$.
- T is the counterclockwise rotation of 45° in R^2 , $\mathbf{v} = (2, 2)$.
- T is the counterclockwise rotation of 120° in R^2 , $\mathbf{v} = (2, 2)$.
- T is the clockwise rotation (θ is negative) of 60° in R^2 , $\mathbf{v} = (1, 2)$.
- T is the clockwise rotation (θ is negative) of 30° in R^2 , $\mathbf{v} = (2, 1)$.
- T is the reflection in the xy -coordinate plane in R^3 : $T(x, y, z) = (x, y, -z), \mathbf{v} = (3, 2, 2)$.
- T is the reflection in the yz -coordinate plane in R^3 : $T(x, y, z) = (-x, y, z), \mathbf{v} = (2, 3, 4)$.
- T is the projection onto the vector $\mathbf{w} = (3, 1)$ in R^2 : $T(\mathbf{v}) = \text{proj}_{\mathbf{w}} \mathbf{v}, \mathbf{v} = (1, 4)$.
- T is the reflection in the vector $\mathbf{w} = (3, 1)$ in R^2 : $T(\mathbf{v}) = 2 \text{proj}_{\mathbf{w}} \mathbf{v} - \mathbf{v}, \mathbf{v} = (1, 4)$.

Finding the Standard Matrix and the Image In Exercises 23–26, (a) find the standard matrix A for the linear transformation T and (b) use A to find the image of the vector $\mathbf{v}$. Use a software program or a graphing utility to verify your result.

- $T(x, y, z) = (2x + 3y - z, 3x - 2z, 2x - y + z), \mathbf{v} = (1, 2, -1)$
- $T(x, y, z) = (x + 2y - 3z, 3x - 5y, y - 3z), \mathbf{v} = (3, 13, 4)$
- $T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_3, x_1 + 2x_2 - x_4, x_4), \mathbf{v} = (1, 0, 1, -1)$
- $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, x_2 - x_1, 2x_3 - x_4, x_1), \mathbf{v} = (0, 1, -1, 1)$

Finding Standard Matrices for Compositions In Exercises 27–30, find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

- $T_1: R^2 \rightarrow R^2, T_1(x, y) = (x - 2y, 2x + 3y)$
 $T_2: R^2 \rightarrow R^2, T_2(x, y) = (y, 0)$
- $T_1: R^3 \rightarrow R^3, T_1(x, y, z) = (x, y, z)$
 $T_2: R^3 \rightarrow R^3, T_2(x, y, z) = (0, x, 0)$
- $T_1: R^2 \rightarrow R^3, T_1(x, y) = (-2x + 3y, x + y, x - 2y)$
 $T_2: R^3 \rightarrow R^2, T_2(x, y, z) = (x - 2y, z + 2x)$
- $T_1: R^2 \rightarrow R^3, T_1(x, y) = (x, y, y)$
 $T_2: R^3 \rightarrow R^2, T_2(x, y, z) = (y, z)$

Finding the Inverse of a Linear Transformation In Exercises 31–36, determine whether the linear transformation is invertible. If it is, find its inverse.

- $T(x, y) = (-4x, 4y)$
- $T(x, y) = (2x, 0)$
- $T(x, y) = (x + y, 3x + 3y)$
- $T(x, y) = (x + y, x - y)$
- $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$
- $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_2, x_3 + x_4, x_3)$

Finding the Image Two Ways In Exercises 37–42, find $T(\mathbf{v})$ by using (a) the standard matrix and (b) the matrix relative to B and B' .

- $T: R^2 \rightarrow R^3, T(x, y) = (x + y, x, y), \mathbf{v} = (5, 4),$
 $B = \{(1, -1), (0, 1)\},$
 $B' = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$
- $T: R^3 \rightarrow R^2, T(x, y, z) = (x - y, y - z), \mathbf{v} = (2, 4, 6),$
 $B = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}, B' = \{(1, 1), (2, 1)\}$
- $T: R^3 \rightarrow R^4, T(x, y, z) = (2x, x + y, y + z, x + z),$
 $\mathbf{v} = (1, -5, 2), B = \{(2, 0, 1), (0, 2, 1), (1, 2, 1)\},$
 $B' = \{(1, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$

40. $T: R^4 \rightarrow R^2$,
 $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, x_4 - x_1)$,
 $\mathbf{v} = (4, -3, 1, 1)$,
 $B = \{(1, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$,
 $B' = \{(1, 1), (2, 0)\}$
41. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + y + z, 2z - x, 2y - z)$,
 $\mathbf{v} = (4, -5, 10)$, $B = \{(2, 0, 1), (0, 2, 1), (1, 2, 1)\}$,
 $B' = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$
42. $T: R^2 \rightarrow R^2$, $T(x, y) = (3x - 13y, x - 4y)$, $\mathbf{v} = (4, 8)$,
 $B = B' = \{(2, 1), (5, 1)\}$
43. Let $T: P_2 \rightarrow P_3$ be the linear transformation $T(p) = xp$.
Find the matrix for T relative to the bases $B = \{1, x, x^2\}$
and $B' = \{1, x, x^2, x^3\}$.
44. Let $T: P_2 \rightarrow P_4$ be the linear transformation $T(p) = x^2p$.
Find the matrix for T relative to the bases $B = \{1, x, x^2\}$
and $B' = \{1, x, x^2, x^3, x^4\}$.
45. **Calculus** Let $B = \{1, x, e^x, xe^x\}$ be a basis for a
subspace W of the space of continuous functions, and
let D_x be the differential operator on W . Find the matrix
for D_x relative to the basis B .
46. **Calculus** Repeat Exercise 45 for $B = \{e^{2x}, xe^{2x}, x^2e^{2x}\}$.
47. **Calculus** Use the matrix from Exercise 45 to evaluate
 $D_x[4x - 3xe^x]$.
48. **Calculus** Use the matrix from Exercise 46 to evaluate
 $D_x[5e^{2x} - 3xe^{2x} + x^2e^{2x}]$.
49. **Calculus** Let $B = \{1, x, x^2, x^3\}$ be a basis for P_3 , and
 $T: P_3 \rightarrow P_4$ be the linear transformation represented by

$$T(x^k) = \int_0^x t^k dt.$$
- (a) Find the matrix A for T with respect to B and the
standard basis for P_4 .
- (b) Use A to integrate $p(x) = 8 - 4x + 3x^3$.

50. CAPSTONE

- (a) Explain how to find the standard matrix for a
linear transformation.
- (b) Explain how to find a composition of linear
transformations.
- (c) Explain how to find the inverse of a linear
transformation.
- (d) Explain how to find the transformation matrix
relative to nonstandard bases.

51. Define $T: M_{2,3} \rightarrow M_{3,2}$ by $T(A) = A^T$.
- (a) Find the matrix for T relative to the standard bases
for $M_{2,3}$ and $M_{3,2}$.
- (b) Show that T is an isomorphism.
- (c) Find the matrix for the inverse of T .

52. Let T be a linear transformation such that $T(\mathbf{v}) = k\mathbf{v}$ for
 $\mathbf{v}$ in R^n . Find the standard matrix for T .

True or False? In Exercises 53 and 54, determine
whether each statement is true or false. If a statement
is true, give a reason or cite an appropriate statement
from the text. If a statement is false, provide an example
that shows the statement is not true in all cases or cite an
appropriate statement from the text.

53. (a) If $T: R^n \rightarrow R^m$ is a linear transformation such that

$$T(\mathbf{e}_1) = [a_{11} \ a_{21} \ \dots \ a_{m1}]^T$$

$$T(\mathbf{e}_2) = [a_{12} \ a_{22} \ \dots \ a_{m2}]^T$$

$$\vdots$$

$$T(\mathbf{e}_n) = [a_{1n} \ a_{2n} \ \dots \ a_{mn}]^T$$

then the $m \times n$ matrix $A = [a_{ij}]$ whose columns
correspond to $T(\mathbf{e}_i)$ and is such that $T(\mathbf{v}) = A\mathbf{v}$ for
every $\mathbf{v}$ in R^n is called the standard matrix for T .
- (b) All linear transformations T have a unique inverse
 T^{-1} .
54. (a) The composition T of linear transformations T_1
and T_2 , represented by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is defined
when the range of T_1 lies within the domain of T_2 .
- (b) In general, the compositions $T_2 \circ T_1$ and $T_1 \circ T_2$
have the same standard matrix A .

55. **Guided Proof** Let $T_1: V \rightarrow V$ and $T_2: V \rightarrow V$ be
one-to-one linear transformations. Prove that the
composition $T = T_2 \circ T_1$ is one-to-one and that T^{-1}
exists and is equal to $T_1^{-1} \circ T_2^{-1}$.

Getting Started: To show that T is one-to-one, use the
definition of a one-to-one transformation and show that
 $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. For the second statement,
you first need to use Theorems 6.8 and 6.12 to show that
 T is invertible, and then show that $T \circ (T_1^{-1} \circ T_2^{-1})$ and
 $(T_1^{-1} \circ T_2^{-1}) \circ T$ are identity transformations.

- (i) Let $T(\mathbf{u}) = T(\mathbf{v})$. Recall that $(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v}))$
for all vectors $\mathbf{v}$. Now use the fact that T_2 and T_1
are one-to-one to conclude that $\mathbf{u} = \mathbf{v}$.
- (ii) Use Theorems 6.8 and 6.12 to show that T_1, T_2 ,
and T are all invertible transformations. So, T_1^{-1}
and T_2^{-1} exist.
- (iii) Form the composition $T' = T_1^{-1} \circ T_2^{-1}$. It is a
linear transformation from V into V . To show
that it is the inverse of T , you need to determine
whether the composition of T with T' on both sides
gives an identity transformation.

56. **Proof** Prove Theorem 6.12.
57. **Writing** Is it always preferable to use the standard
basis for R^n ? Discuss the advantages and disadvantages
of using different bases.
58. **Writing** Look back at Theorem 4.19 and rephrase it
in terms of what you have learned in this chapter.

6.4 Transition Matrices and Similarity

- Find and use a matrix for a linear transformation.
- Show that two matrices are similar and use the properties of similar matrices.

THE MATRIX FOR A LINEAR TRANSFORMATION

In Section 6.3, you saw that the matrix for a linear transformation $T: V \rightarrow V$ depends on the basis for V . In other words, the matrix for T relative to a basis B is different from the matrix for T relative to another basis B' .

A classical problem in linear algebra is determining whether it is possible to find a basis B such that the matrix for T relative to B is diagonal. The solution of this problem is discussed in Chapter 7. This section lays a foundation for solving the problem. You will see how the matrices for a linear transformation relative to two different bases are related. In this section, A, A', P , and P^{-1} represent the four square matrices listed below.

1. Matrix for T relative to B : A
2. Matrix for T relative to B' : A'
3. Transition matrix from B' to B : P
4. Transition matrix from B to B' : P^{-1}

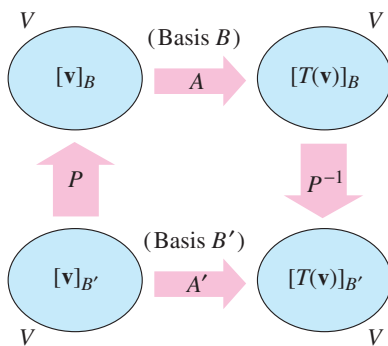


Figure 6.4

Note that in Figure 6.4, there are two ways to get from the coordinate matrix $[v]_{B'}$ to the coordinate matrix $[T(v)]_{B'}$. One way is direct, using the matrix A' to obtain

$$A'[v]_{B'} = [T(v)]_{B'}.$$

The other way is indirect, using the matrices P, A , and P^{-1} to obtain

$$P^{-1}AP[v]_{B'} = [T(v)]_{B'}.$$

This implies that $A' = P^{-1}AP$. Example 1 demonstrates this relationship.

EXAMPLE 1

Finding a Matrix for a Linear Transformation

Find the matrix A' for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$, relative to the basis $B' = \{(1, 0), (1, 1)\}$.

SOLUTION

The standard matrix for T is $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$.

Furthermore, using the techniques of Section 4.7, the transition matrix from B' to the standard basis $B = \{(1, 0), (0, 1)\}$ is

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The inverse of this matrix is the transition matrix from B to B' ,

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

So, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$



In Example 1, the basis B is the standard basis for R^2 . In the next example, both B and B' are nonstandard bases.

EXAMPLE 2**Finding a Matrix for a Linear Transformation**

Let $B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ be bases for R^2 , and let

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$$

be the matrix for $T: R^2 \rightarrow R^2$ relative to B . Find A' , the matrix of T relative to B' .

SOLUTION

In Example 5 in Section 4.7, you found that $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$.

So, the matrix of T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}.$$

Figure 6.4 should help you to remember the roles of the matrices A , A' , P , and P^{-1} .

EXAMPLE 3**Using a Matrix for a Linear Transformation**

For the linear transformation $T: R^2 \rightarrow R^2$ from Example 2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$ for the vector $\mathbf{v}$ whose coordinate matrix is $[\mathbf{v}]_{B'} = [-3 \ -1]^T$.

SOLUTION

To find $[\mathbf{v}]_B$, use the transition matrix P from B' to B .

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

To find $[T(\mathbf{v})]_B$, multiply $[\mathbf{v}]_B$ on the left by the matrix A to obtain

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}.$$

To find $[T(\mathbf{v})]_{B'}$, multiply $[T(\mathbf{v})]_B$ on the left by P^{-1} to obtain

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

or multiply $[\mathbf{v}]_{B'}$ on the left by A' to obtain

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}.$$

REMARK

It is instructive to note that the rule $T(x, y) = (x - \frac{3}{2}y, 2x + 4y)$ represents the transformation T in Examples 2 and 3. Verify the results of Example 3 by showing that $\mathbf{v} = (1, -4)$ and $T(\mathbf{v}) = (7, -14)$.

**LINEAR ALGEBRA APPLIED**

A **Leslie matrix**, named after British mathematician Patrick H. Leslie (1900–1974), can be used to find the age and growth distributions of a population over time. The entries in the first row of an $n \times n$ Leslie matrix L are the average numbers of offspring per member for each of n age classes. The entries in subsequent rows are p_i in row $i + 1$, column i and 0 elsewhere, where p_i is the probability that an i th age class member will survive to become an $(i + 1)$ th age class member. If $\mathbf{x}_j$ is the age distribution vector for the j th time period, then the age distribution vector for the $(j + 1)$ th time period can be found using the linear transformation $\mathbf{x}_{j+1} = L\mathbf{x}_j$. You will study population growth models using Leslie matrices in more detail in Section 7.4.

Rich Lindie/Shutterstock.com

SIMILAR MATRICES

Two square matrices A and A' that are related by an equation $A' = P^{-1}AP$ are called **similar** matrices, as stated in the next definition.

Definition of Similar Matrices

For square matrices A and A' of order n , A' is **similar** to A when there exists an invertible matrix P such that $A' = P^{-1}AP$.

If A' is similar to A , then it is also true that A is similar to A' , as stated in the next theorem. So, it makes sense to simply say that **A and A' are similar**.

THEOREM 6.13 Properties of Similar Matrices

Let A , B , and C be square matrices of order n . Then the properties below are true.

1. A is similar to A .
2. If A is similar to B , then B is similar to A .
3. If A is similar to B and B is similar to C , then A is similar to C .

PROOF

The first property follows from the fact that $A = I_n A I_n$. To prove the second property, write

$$\begin{aligned} A &= P^{-1}BP \\ PAP^{-1} &= P(P^{-1}BP)P^{-1} \\ PAP^{-1} &= B \\ Q^{-1}AQ &= B, \text{ where } Q = P^{-1}. \end{aligned}$$

The proof of the third property is left to you. (See Exercise 33.) 

From the definition of similarity, it follows that any two matrices that represent the same linear transformation $T: V \rightarrow V$ with respect to different bases must be similar.

EXAMPLE 4 Similar Matrices

See LarsonLinearAlgebra.com for an interactive version of this type of example.

- a. From Example 1, the matrices

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

are similar because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- b. From Example 2, the matrices

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

are similar because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$. 

You have seen that the matrix for a linear transformation $T: V \rightarrow V$ depends on the basis used for V . This observation leads naturally to the question: What choice of basis will make the matrix for T as simple as possible? Is it always the *standard* basis? Not necessarily, as the next example demonstrates.

EXAMPLE 5**A Comparison of Two Matrices for a Linear Transformation**

Let

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

be the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for T relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.

SOLUTION

The transition matrix from B' to the standard basis has columns consisting of the vectors in B' ,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and it follows that

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the matrix for T relative to B' is

$$\begin{aligned} A' &= P^{-1}AP \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

Note that matrix A' is diagonal. 

Diagonal matrices have many computational advantages over nondiagonal matrices. For example, for the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

the k th power of D is

$$D^k = \begin{bmatrix} d_1^k & 0 & \dots & 0 \\ 0 & d_2^k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n^k \end{bmatrix}.$$

Also, a diagonal matrix is its own transpose. Moreover, if all of the main diagonal entries of a diagonal matrix are nonzero, then the inverse of the matrix is also a diagonal matrix, whose main diagonal entries are the reciprocals of corresponding entries in the original matrix. With such computational advantages, it is important to find ways (if possible) to choose a basis for V such that the transformation matrix is diagonal, as it is in Example 5. You will pursue this problem in the next chapter.

6.4 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding a Matrix for a Linear Transformation In Exercises 1–12, find the matrix A' for T relative to the basis B' .

- $T: R^2 \rightarrow R^2$, $T(x, y) = (2x - y, y - x)$,
 $B' = \{(1, -2), (0, 3)\}$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (2x + y, x - 2y)$,
 $B' = \{(1, 2), (0, 4)\}$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (x + y, 4y)$,
 $B' = \{(-4, 1), (1, -1)\}$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (x - 2y, 4x)$,
 $B' = \{(-2, 1), (-1, 1)\}$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (-3x + y, 3x - y)$,
 $B' = \{(1, -1), (-1, 5)\}$
- $T: R^2 \rightarrow R^2$, $T(x, y) = (5x + 4y, 4x + 5y)$,
 $B' = \{(12, -13), (13, -12)\}$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x, y, z)$,
 $B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (0, 0, 0)$,
 $B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (y + z, x + z, x + y)$,
 $B' = \{(5, 0, -1), (-3, 2, -1), (4, -6, 5)\}$
- $T: R^3 \rightarrow R^3$, $T(x, y, z) = (-x, x - y, y - z)$,
 $B' = \{(0, -1, 2), (-2, 0, 3), (1, 3, 0)\}$
- $T: R^3 \rightarrow R^3$,
 $T(x, y, z) = (x - y + 2z, 2x + y - z, x + 2y + z)$,
 $B' = \{(1, 0, 1), (0, 2, 2), (1, 2, 0)\}$
- $T: R^3 \rightarrow R^3$,
 $T(x, y, z) = (x, x + 2y, x + y + 3z)$,
 $B' = \{(1, -1, 0), (0, 0, 1), (0, 1, -1)\}$
- Let $B = \{(1, 3), (-2, -2)\}$ and $B' = \{(-12, 0), (-4, 4)\}$ be bases for R^2 , and let
$$A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$
be the matrix for $T: R^2 \rightarrow R^2$ relative to B .
(a) Find the transition matrix P from B' to B .
(b) Use the matrices P and A to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where $[\mathbf{v}]_{B'} = [-1 \ 2]^T$.
(c) Find P^{-1} and A' (the matrix for T relative to B').
(d) Find $[T(\mathbf{v})]_{B'}$ two ways.
- Repeat Exercise 13 for $B = \{(1, 1), (-2, 3)\}$, $B' = \{(1, -1), (0, 1)\}$, and $[\mathbf{v}]_{B'} = [1 \ -3]^T$.
(Use matrix A in Exercise 13.)

15. Let $B = \{(1, 2), (-1, -1)\}$ and $B' = \{(-4, 1), (0, 2)\}$ be bases for R^2 , and let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

be the matrix for $T: R^2 \rightarrow R^2$ relative to B .

- (a) Find the transition matrix P from B' to B .
(b) Use the matrices P and A to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where $[\mathbf{v}]_{B'} = [-1 \ 4]^T$.
(c) Find P^{-1} and A' (the matrix for T relative to B').
(d) Find $[T(\mathbf{v})]_{B'}$ two ways.
16. Repeat Exercise 15 for $B = \{(1, -1), (-2, 1)\}$, $B' = \{(-1, 1), (1, 2)\}$, and $[\mathbf{v}]_{B'} = [1 \ -4]^T$.
(Use matrix A in Exercise 15.)
17. Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be bases for R^3 , and let

$$A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$$

be the matrix for $T: R^3 \rightarrow R^3$ relative to B .

- (a) Find the transition matrix P from B' to B .
(b) Use the matrices P and A to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where $[\mathbf{v}]_{B'} = [1 \ 0 \ -1]^T$.
(c) Find P^{-1} and A' (the matrix for T relative to B').
(d) Find $[T(\mathbf{v})]_{B'}$ two ways.
18. Repeat Exercise 17 for
 $B = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$,
 $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and
 $[\mathbf{v}]_{B'} = [2 \ 1 \ 1]^T$.
(Use matrix A in Exercise 17.)

Similar Matrices In Exercises 19–22, use the matrix P to show that the matrices A and A' are similar.

19. $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$
20. $P = A = A' = \begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$
21. $P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 10 & 0 \\ 8 & 4 & 0 \\ 0 & 9 & 6 \end{bmatrix}$, $A' = \begin{bmatrix} 5 & 8 & 0 \\ 10 & 4 & 0 \\ 0 & 12 & 6 \end{bmatrix}$
22. $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A' = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Diagonal Matrix for a Linear Transformation In Exercises 23 and 24, let A be the matrix for $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ relative to the standard basis. Find the diagonal matrix A' for T relative to the basis B' .

$$23. A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B' = \{(-1, 1, 0), (2, 1, 0), (0, 0, 1)\}$$

$$24. A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix},$$

$$B' = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$$

25. **Proof** Prove that if A and B are similar matrices, then $|A| = |B|$.

Is the converse true?

26. Illustrate the result of Exercise 25 using the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 11 & 7 & 10 \\ 10 & 8 & 10 \\ -18 & -12 & -17 \end{bmatrix},$$

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 2 \\ 1 & 2 & -3 \end{bmatrix},$$

where $B = P^{-1}AP$.

27. **Proof** Prove that if A and B are similar matrices, then there exists a matrix P such that $B^k = P^{-1}A^kP$.

28. Use the result of Exercise 27 to find B^4 , where

$$B = P^{-1}AP$$

for the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix},$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, P^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

29. Determine all $n \times n$ matrices that are similar to I_n .

30. **Proof** Prove that if A is an idempotent matrix and B is similar to A , then B is idempotent. (Recall that an $n \times n$ matrix A is idempotent when $A = A^2$.)

31. **Proof** Let A be an $n \times n$ matrix such that $A^2 = O$. Prove that if B is similar to A , then $B^2 = O$.

32. **Proof** Consider the matrix equation $B = P^{-1}AP$. Prove that if $A\mathbf{x} = \mathbf{x}$, then $PBP^{-1}\mathbf{x} = \mathbf{x}$.

33. **Proof** Prove Property 3 of Theorem 6.13: For square matrices A , B , and C of order n , if A is similar to B and B is similar to C , then A is similar to C .

34. **Writing** Explain why two similar matrices have the same rank.

35. **Proof** Prove that if A and B are similar matrices, then A^T and B^T are similar matrices.

36. **Proof** Prove that if A and B are similar matrices and A is nonsingular, then B is also nonsingular and A^{-1} and B^{-1} are similar matrices.

37. **Proof** Let $A = CD$, where C and D are $n \times n$ matrices and C is invertible. Prove that the matrix product DC is similar to A .

38. **Proof** Let $B = P^{-1}AP$, where $A = [a_{ij}]$, $P = [p_{ij}]$, and B is a diagonal matrix with main diagonal entries $b_{11}, b_{22}, \dots, b_{nn}$. Prove that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix} = b_{ii} \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix}$$

for $i = 1, 2, \dots, n$.

39. **Writing** Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for the vector space V , let B' be the standard basis, and consider the identity transformation $I: V \rightarrow V$. What can you say about the matrix for I relative to B' relative to B' ? when the domain has the basis B and the range has the basis B' ?

40. CAPSTONE

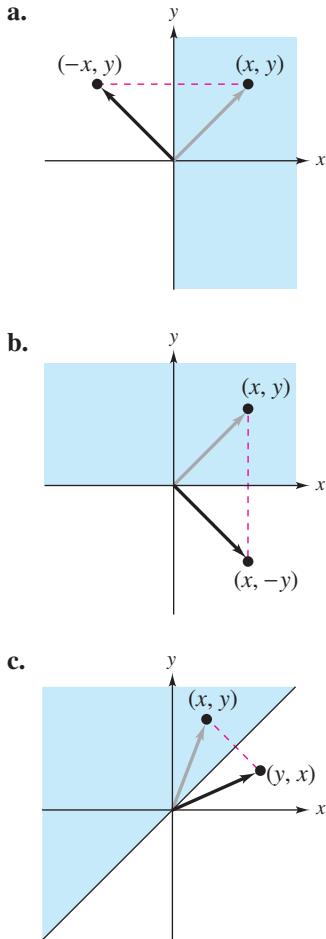
- Consider two bases B and B' for a vector space V and the matrix A for the linear transformation $T: V \rightarrow V$ relative to B . Explain how to obtain the coordinate matrix $[T(\mathbf{v})]_{B'}$ from the coordinate matrix $[\mathbf{v}]_{B'}$, where $\mathbf{v}$ is a vector in V .
- Explain how to determine whether two square matrices A and A' of order n are similar.

True or False? In Exercises 41 and 42, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- The matrix for a linear transformation A' relative to the basis B' is equal to the product $P^{-1}AP$, where P^{-1} is the transition matrix from B to B' , A is the matrix for the linear transformation relative to basis B , and P is the transition matrix from B' to B .
 - Two matrices that represent the same linear transformation $T: V \rightarrow V$ with respect to different bases are not necessarily similar.
- The matrix for a linear transformation A relative to the basis B is equal to the product $PA'P^{-1}$, where P is the transition matrix from B' to B , A' is the matrix for the linear transformation relative to basis B' , and P^{-1} is the transition matrix from B to B' .
 - The standard basis for $\mathbb{R}^n$ will always make the coordinate matrix for the linear transformation T the simplest matrix possible.

6.5 Applications of Linear Transformations

- Identify linear transformations defined by reflections, expansions, contractions, or shears in R^2 .
- Use a linear transformation to rotate a figure in R^3 .



Reflections in R^2
Figure 6.5

THE GEOMETRY OF LINEAR TRANSFORMATIONS IN R^2

The first part of this section gives geometric interpretations of linear transformations represented by 2×2 elementary matrices. After a summary of the various types of 2×2 elementary matrices are examples that examine each type of matrix in more detail.

Elementary Matrices for Linear Transformations in R^2

Reflection in y-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal Expansion ($k > 1$) or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion ($k > 1$) or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal Shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

EXAMPLE 1

Reflections in R^2

The transformations below are **reflections**. These have the effect of mapping a point in the xy -plane to its “mirror image” with respect to one of the coordinate axes or the line $y = x$, as shown in Figure 6.5.

a. Reflection in the y-axis:

$$T(x, y) = (-x, y)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

b. Reflection in the x-axis:

$$T(x, y) = (x, -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

c. Reflection in the line $y = x$:

$$T(x, y) = (y, x)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

EXAMPLE 2**Expansions and Contractions in $\mathbb{R}^2$**

The transformations below are **expansions** or **contractions**, depending on the value of the positive scalar k .

a. Horizontal expansions and contractions:

$$T(x, y) = (kx, y)$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

b. Vertical expansions and contractions:

$$T(x, y) = (x, ky)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$

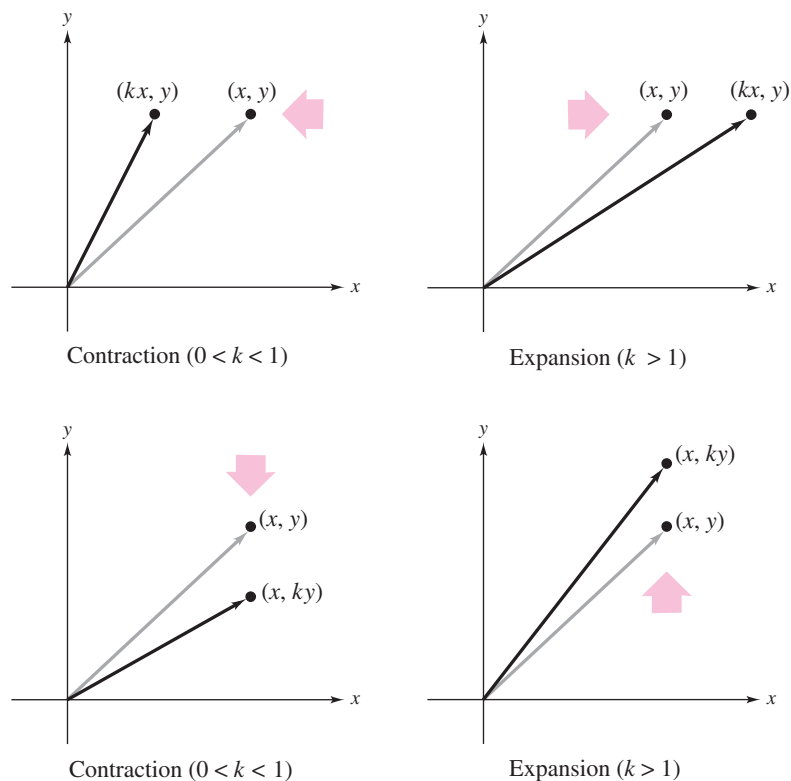
Note in the figures below that the distance the point (x, y) moves by a contraction or an expansion is proportional to its x - or y -coordinate. For example, under the transformation represented by

$$T(x, y) = (2x, y)$$

the point $(1, 3)$ would move one unit to the right, but the point $(4, 3)$ would move four units to the right. Under the transformation represented by

$$T(x, y) = \left(x, \frac{1}{2}y\right)$$

the point $(1, 4)$ would move two units down, but the point $(1, 2)$ would move one unit down.



Another type of linear transformation in $\mathbb{R}^2$ corresponding to an elementary matrix is a *shear*, as described in Example 3.

EXAMPLE 3**Shears in R^2**

The transformations below are **shears**.

$$T(x, y) = (x + ky, y)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

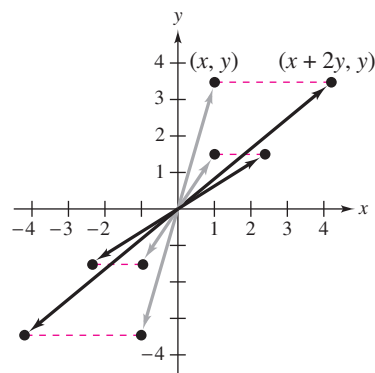
$$T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

- a.** A horizontal shear represented by

$$T(x, y) = (x + 2y, y)$$

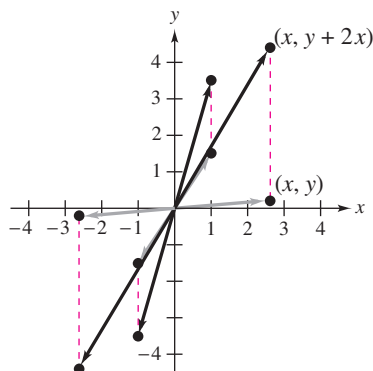
is shown at the right. Under this transformation, points in the upper half-plane “shear” to the right by amounts proportional to their y -coordinates. Points in the lower half-plane “shear” to the left by amounts proportional to the absolute values of their y -coordinates. Points on the x -axis do not move by this transformation.



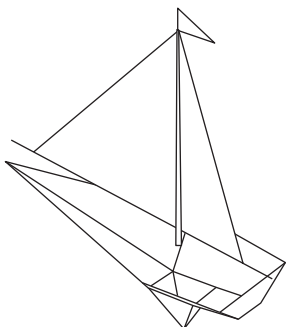
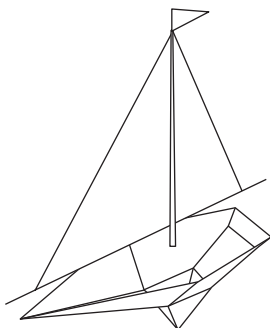
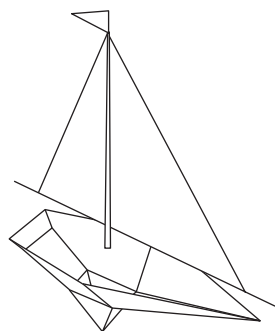
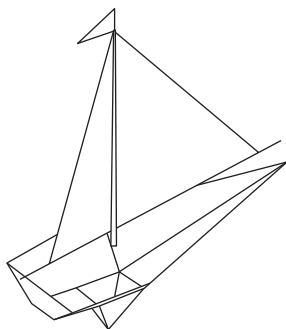
- b.** A vertical shear represented by

$$T(x, y) = (x, y + 2x)$$

is shown below. Here, points in the right half-plane “shear” upward by amounts proportional to their x -coordinates. Points in the left half-plane “shear” downward by amounts proportional to the absolute values of their x -coordinates. Points on the y -axis do not move.

**LINEAR
ALGEBRA
APPLIED**

The use of computer graphics is common in many fields. By using graphics software, a designer can “see” an object before it is physically created. Linear transformations can be useful in computer graphics. To illustrate, consider a simplified example. Only 23 points in R^3 were used to generate images of the toy boat shown at the left. Most graphics software can use such minimal information to generate views of an image from any perspective, as well as color, shade, and render as appropriate. Linear transformations, specifically those that produce rotations in R^3 , can represent the different views. The remainder of this section discusses rotation in R^3 .



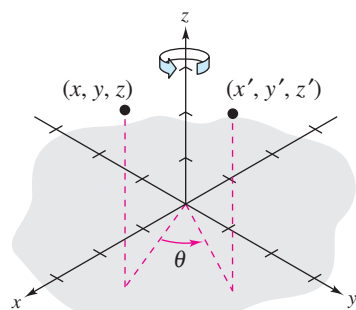


Figure 6.6

ROTATION IN R^3

In Example 7 in Section 6.1, you saw how a linear transformation can be used to rotate figures in R^2 . Here you will see how linear transformations can be used to rotate figures in R^3 .

Say you want to rotate the point (x, y, z) counterclockwise about the z -axis through an angle θ , as shown in Figure 6.6. Letting the coordinates of the rotated point be (x', y', z') , you have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ z \end{bmatrix}.$$

Example 4 uses this matrix to rotate a figure in three-dimensional space.

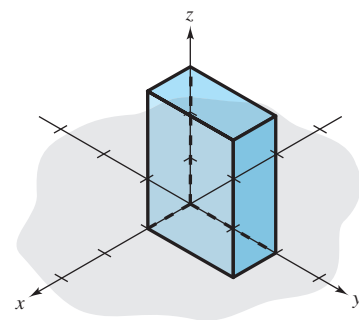
EXAMPLE 4

Rotation About the z -Axis

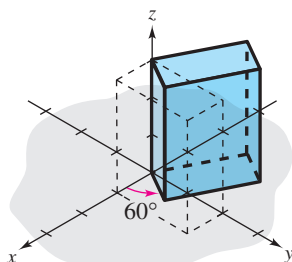
The eight vertices of the rectangular prism shown at the right are

$$\begin{array}{ll} V_1(0, 0, 0) & V_2(1, 0, 0) \\ V_3(1, 2, 0) & V_4(0, 2, 0) \\ V_5(0, 0, 3) & V_6(1, 0, 3) \\ V_7(1, 2, 3) & V_8(0, 2, 3). \end{array}$$

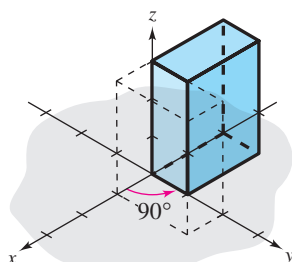
Find the coordinates of the vertices after the prism is rotated counterclockwise about the z -axis through (a) $\theta = 60^\circ$, (b) $\theta = 90^\circ$, and (c) $\theta = 120^\circ$.



a.



b.



c.

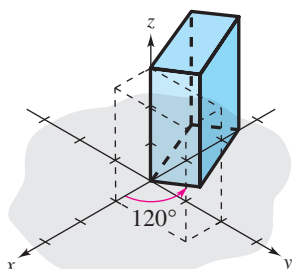


Figure 6.7

SOLUTION

a. The matrix that yields a rotation of 60° is

$$A = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by the column vectors corresponding to each vertex produces the rotated vertices listed below.

$$\begin{array}{llll} V_1'(0, 0, 0) & V_2'(0.5, 0.87, 0) & V_3'(-1.23, 1.87, 0) & V_4'(-1.73, 1, 0) \\ V_5'(0, 0, 3) & V_6'(0.5, 0.87, 3) & V_7'(-1.23, 1.87, 3) & V_8'(-1.73, 1, 3) \end{array}$$

Figure 6.7(a) shows a graph of the rotated prism.

b. The matrix that yields a rotation of 90° is

$$A = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.7(b) shows a graph of the rotated prism.

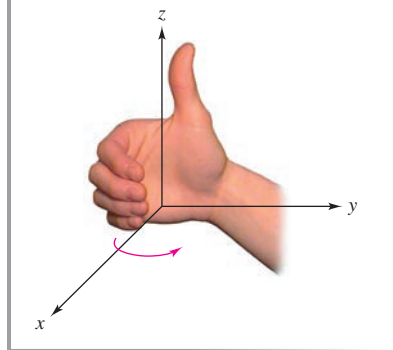
c. The matrix that yields a rotation of 120° is

$$A = \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ & 0 \\ \sin 120^\circ & \cos 120^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.7(c) shows a graph of the rotated prism.

REMARK

To illustrate the right-hand rule, imagine the thumb of your right hand pointing in the positive direction of an axis. The cupped fingers will point in the direction of counterclockwise rotation. The figure below shows counterclockwise rotation about the z -axis.



Example 4 uses matrices to perform rotations about the z -axis. Similarly, you can use matrices to rotate figures about the x - or y -axis. A summary of all three types of rotations is below.

Rotation About the x -Axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

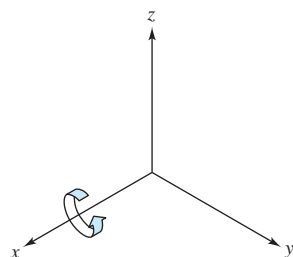
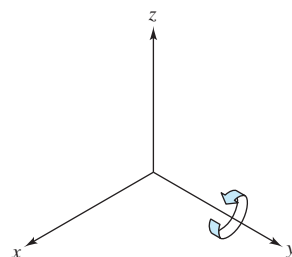
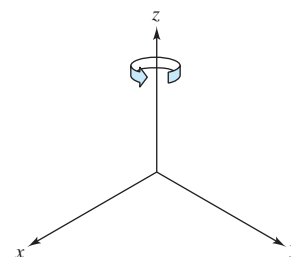
Rotation About the y -Axis

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

Rotation About the z -Axis

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In each case, the rotation is oriented counterclockwise (using the “right-hand rule”) relative to the specified axis, as shown below.

Rotation about x -axisRotation about y -axisRotation about z -axis**EXAMPLE 5****Rotation About the x -Axis and y -Axis**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

- a. The matrix that yields a rotation of 90° about the x -axis is

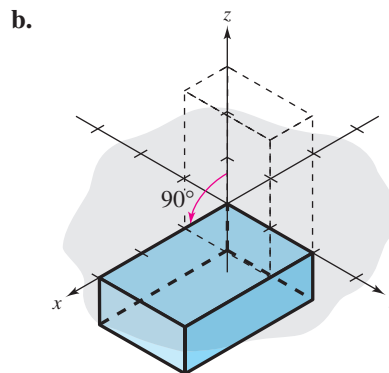
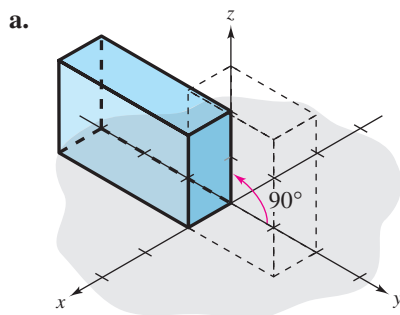
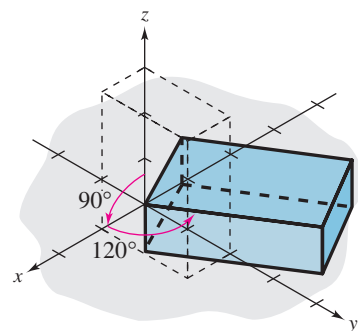
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Figure 6.8(a) shows the prism from Example 4 rotated 90° about the x -axis.

- b. The matrix that yields a rotation of 90° about the y -axis is

$$A = \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Figure 6.8(b) shows the prism from Example 4 rotated 90° about the y -axis.

**Figure 6.8****Figure 6.9**

Rotations about the coordinate axes can be combined to produce any desired view of a figure. For example, Figure 6.9 shows the prism from Example 4 rotated 90° about the y -axis and then 120° about the z -axis.

6.5 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



- Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a reflection in the x -axis. Find the image of each vector.
 - $(3, 5)$
 - $(2, -1)$
 - $(a, 0)$
 - $(0, b)$
 - $(-c, d)$
 - $(f, -g)$
- Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a reflection in the y -axis. Find the image of each vector.
 - $(5, 2)$
 - $(-1, -6)$
 - $(a, 0)$
 - $(0, b)$
 - $(c, -d)$
 - (f, g)
- Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a reflection in the line $y = x$. Find the image of each vector.
 - $(0, 1)$
 - $(-1, 3)$
 - $(a, 0)$
 - $(0, b)$
 - $(-c, d)$
 - $(f, -g)$
- Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a reflection in the line $y = -x$. Find the image of each vector.
 - $(-1, 2)$
 - $(2, 3)$
 - $(a, 0)$
 - $(0, b)$
 - $(e, -d)$
 - $(-f, g)$
- Let $T(1, 0) = (2, 0)$ and $T(0, 1) = (0, 1)$.
 - Determine $T(x, y)$ for any (x, y) .
 - Give a geometric description of T .
- Let $T(1, 0) = (1, 1)$ and $T(0, 1) = (0, 1)$.
 - Determine $T(x, y)$ for any (x, y) .
 - Give a geometric description of T .

Identifying and Representing a Transformation In Exercises 7–14, (a) identify the transformation, and (b) graphically represent the transformation for an arbitrary vector in $\mathbb{R}^2$.

- $T(x, y) = (x, y/2)$
- $T(x, y) = (x/4, y)$
- $T(x, y) = (12x, y)$
- $T(x, y) = (x, 3y)$
- $T(x, y) = (x + 3y, y)$
- $T(x, y) = (x + 4y, y)$
- $T(x, y) = (x, 5x + y)$
- $T(x, y) = (x, 9x + y)$

Finding Fixed Points of a Linear Transformation In Exercises 15–22, find all fixed points of the linear transformation. Recall that the vector $\mathbf{v}$ is a fixed point of T when $T(\mathbf{v}) = \mathbf{v}$.

- A reflection in the y -axis
- A reflection in the x -axis
- A reflection in the line $y = x$
- A reflection in the line $y = -x$
- A vertical contraction
- A horizontal expansion
- A horizontal shear
- A vertical shear

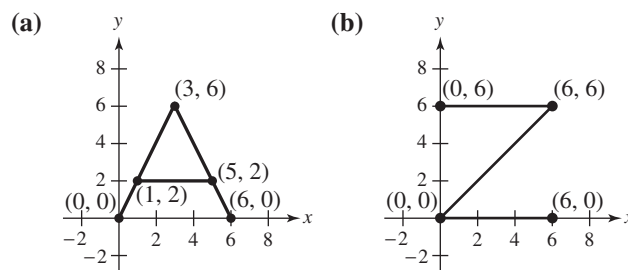
Sketching an Image of the Unit Square In Exercises 23–30, sketch the image of the unit square [a square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$] under the specified transformation.

- T is a reflection in the x -axis.
- T is a reflection in the line $y = x$.
- T is the contraction represented by $T(x, y) = (x/2, y)$.
- T is the contraction represented by $T(x, y) = (x, y/4)$.
- T is the expansion represented by $T(x, y) = (x, 3y)$.
- T is the expansion represented by $T(x, y) = (5x, y)$.
- T is the shear represented by $T(x, y) = (x + 2y, y)$.
- T is the shear represented by $T(x, y) = (x, y + 3x)$.

Sketching an Image of a Rectangle In Exercises 31–38, sketch the image of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, 2)$, and $(0, 2)$ under the specified transformation.

- T is a reflection in the y -axis.
- T is a reflection in the line $y = x$.
- T is the contraction represented by $T(x, y) = (x/3, y)$.
- T is the contraction represented by $T(x, y) = (x, y/2)$.
- T is the expansion represented by $T(x, y) = (x, 6y)$.
- T is the expansion represented by $T(x, y) = (2x, y)$.
- T is the shear represented by $T(x, y) = (x + y, y)$.
- T is the shear represented by $T(x, y) = (x, y + 2x)$.

Sketching an Image of a Figure In Exercises 39–44, sketch each of the images under the specified transformation.



- T is a reflection in the x -axis.
- T is a reflection in the line $y = x$.
- T is the shear represented by $T(x, y) = (x + y, y)$.
- T is the shear represented by $T(x, y) = (x, x + y)$.
- T is the expansion and contraction represented by $T(x, y) = (2x, \frac{1}{2}y)$.
- T is the expansion and contraction represented by $T(x, y) = (\frac{1}{2}x, 2y)$.

Giving a Geometric Description In Exercises 45–50, give a geometric description of the linear transformation defined by the elementary matrix.

$$45. A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad 46. A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$47. A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \quad 48. A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$49. A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad 50. A = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$$

Giving a Geometric Description In Exercises 51 and 52, give a geometric description of the linear transformation defined by the matrix product.

$$51. A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$52. A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

53. The linear transformation defined by a diagonal matrix with positive main diagonal elements is called a **magnification**. Find the images of $(1, 0)$, $(0, 1)$, and $(2, 2)$ under the linear transformation defined by A and graphically interpret your result.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

54. CAPSTONE Describe the transformation defined by each matrix. Assume k and θ are positive scalars.

$$(a) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(d) \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, k > 1$$

$$(e) \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, 0 < k < 1$$

$$(f) \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, k > 1$$

$$(g) \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, 0 < k < 1$$

$$(h) \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$(i) \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$(j) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$(k) \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$(l) \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding a Matrix to Produce a Rotation In Exercises 55–58, find the matrix that produces the rotation.

55. 30° about the z -axis 56. 60° about the x -axis

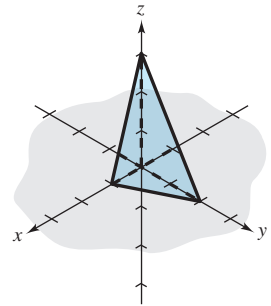
57. 120° about the x -axis 58. 60° about the y -axis

Finding the Image of a Vector In Exercises 59–62, find the image of the vector $(1, 1, 1)$ for the rotation.

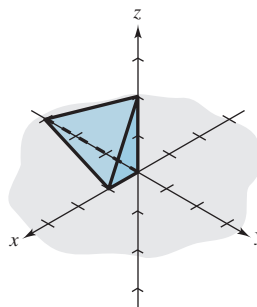
59. 30° about the z -axis 60. 60° about the x -axis

61. 120° about the x -axis 62. 60° about the y -axis

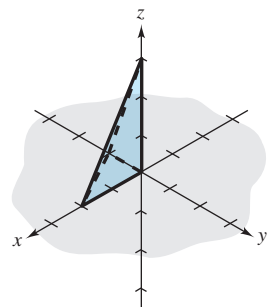
Determining a Rotation In Exercises 63–68, determine which single counterclockwise rotation about the x -, y -, or z -axis produces the rotated tetrahedron. The figure at the right shows the tetrahedron before rotation.



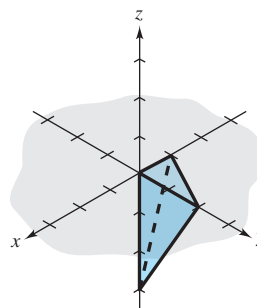
63.



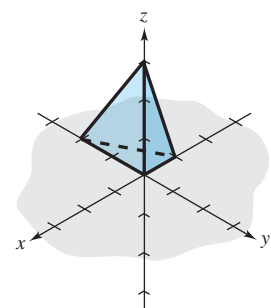
64.



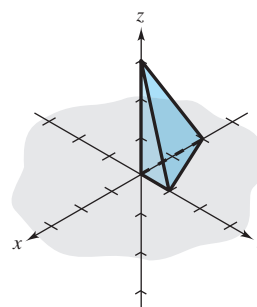
65.



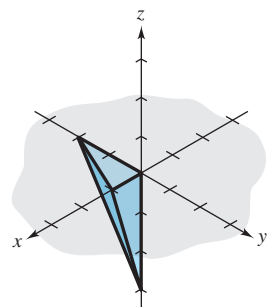
66.



67.



68.



Determining a Matrix to Produce a Pair of Rotations In Exercises 69–72, determine the matrix that produces the pair of rotations. Then find the image of the vector $(1, 1, 1)$ under these rotations.

69. 90° about the x -axis and then 90° about the y -axis

70. 30° about the z -axis and then 60° about the y -axis

71. 45° about the z -axis and then 135° about the x -axis

72. 120° about the x -axis and then 135° about the z -axis

6 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding an Image and a Preimage In Exercises 1–6, find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$ for the linear transformation.

1. $T: R^2 \rightarrow R^2$, $T(v_1, v_2) = (v_1, v_1 + 2v_2)$, $\mathbf{v} = (2, -3)$, $\mathbf{w} = (4, 12)$
2. $T: R^2 \rightarrow R^2$, $T(v_1, v_2) = (v_1 + v_2, 2v_2)$, $\mathbf{v} = (4, -1)$, $\mathbf{w} = (8, 4)$
3. $T: R^3 \rightarrow R^3$, $T(v_1, v_2, v_3) = (0, v_1 + v_2, v_2 + v_3)$, $\mathbf{v} = (-3, 2, 5)$, $\mathbf{w} = (0, 2, 5)$
4. $T: R^3 \rightarrow R^3$, $T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3, v_3)$, $\mathbf{v} = (-2, 1, 2)$, $\mathbf{w} = (0, 1, 2)$
5. $T: R^2 \rightarrow R^3$, $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2, 2v_1 + 3v_2)$, $\mathbf{v} = (2, -3)$, $\mathbf{w} = (1, -3, 4)$
6. $T: R^2 \rightarrow R$, $T(v_1, v_2) = (2v_1 - v_2)$, $\mathbf{v} = (2, -3)$, $\mathbf{w} = 4$

Linear Transformations and Standard Matrices In Exercises 7–18, determine whether the function is a linear transformation. If it is, find its standard matrix A .

7. $T: R \rightarrow R^2$, $T(x) = (x, x + 2)$
8. $T: R^2 \rightarrow R$, $T(x_1, x_2) = (x_1 + x_2)$
9. $T: R^2 \rightarrow R^2$, $T(x_1, x_2) = (x_1 + 2x_2, -x_1 - x_2)$
10. $T: R^2 \rightarrow R^2$, $T(x_1, x_2) = (x_1 + 3, x_2)$
11. $T: R^2 \rightarrow R^2$, $T(x, y) = (x - 2y, 2y - x)$
12. $T: R^2 \rightarrow R^2$, $T(x, y) = (x + y, y)$
13. $T: R^2 \rightarrow R^2$, $T(x, y) = (x + h, y + k)$, $h \neq 0$ or $k \neq 0$ (translation in R^2)
14. $T: R^2 \rightarrow R^2$, $T(x, y) = (|x|, |y|)$
15. $T: R^3 \rightarrow R^3$, $T(x_1, x_2, x_3) = (x_1 + x_2, 2, x_3 - x_1)$
16. $T: R^3 \rightarrow R^3$, $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$
17. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (z, y, x)$
18. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x, 0, -y)$
19. Let T be a linear transformation from R^2 into R^2 such that $T(2, 0) = (1, 1)$ and $T(0, 3) = (3, 3)$. Find $T(1, 1)$ and $T(0, 1)$.
20. Let T be a linear transformation from R^3 into R such that $T(1, 1, 1) = 1$, $T(1, 1, 0) = 2$, and $T(1, 0, 0) = 3$. Find $T(0, 1, 1)$.
21. Let T be a linear transformation from R^2 into R^2 such that $T(4, -2) = (2, -2)$ and $T(3, 3) = (-3, 3)$. Find $T(-7, 2)$.
22. Let T be a linear transformation from R^2 into R^2 such that $T(1, -1) = (2, -3)$ and $T(0, 2) = (0, 8)$. Find $T(2, 4)$.

Linear Transformation Given by a Matrix In Exercises 23–28, define the linear transformation $T: R^n \rightarrow R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Use the matrix A to (a) determine the dimensions of R^n and R^m , (b) find the image of $\mathbf{v}$, and (c) find the preimage of $\mathbf{w}$.

23. $A = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$, $\mathbf{v} = (6, 1, 1)$, $\mathbf{w} = (3, 5)$
24. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = (5, 2, 2)$, $\mathbf{w} = (4, 2)$
25. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = (2, 1, -5)$, $\mathbf{w} = (6, 4, 2)$
26. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{v} = (8, 4)$, $\mathbf{w} = (5, 2)$
27. $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$, $\mathbf{v} = (2, 2)$, $\mathbf{w} = (4, -5, 0)$
28. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}$, $\mathbf{v} = (3, 5)$, $\mathbf{w} = (5, 2, -1)$

29. Use the standard matrix for counterclockwise rotation in R^2 to rotate the triangle with vertices $(3, 5)$, $(5, 3)$, and $(3, 0)$ counterclockwise 90° about the origin. Graph the triangles.
30. Rotate the triangle in Exercise 29 counterclockwise 90° about the point $(5, 3)$. Graph the triangles.

Finding the Kernel and Range In Exercises 31–34, find (a) $\ker(T)$ and (b) $\text{range}(T)$.

31. $T: R^4 \rightarrow R^3$,
 $T(w, x, y, z) = (2w + 4x + 6y + 5z,$
 $-w - 2x + 2y, 8y + 4z)$
32. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + 2y, y + 2z, z + 2x)$
33. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x, y, z + 3y)$
34. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + y, y + z, x - z)$

Finding the Kernel, Nullity, Range, and Rank In Exercises 35–38, define the linear transformation T by $T(\mathbf{v}) = A\mathbf{v}$. Find (a) $\ker(T)$, (b) $\text{nullity}(T)$, (c) $\text{range}(T)$, and (d) $\text{rank}(T)$.

35. $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$
36. $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ -2 & 2 \end{bmatrix}$
37. $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$
38. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

39. For $T: R^5 \rightarrow R^3$ and $\text{nullity}(T) = 2$, find $\text{rank}(T)$.
 40. For $T: P_5 \rightarrow P_3$ and $\text{nullity}(T) = 4$, find $\text{rank}(T)$.
 41. For $T: P_4 \rightarrow R^5$ and $\text{rank}(T) = 3$, find $\text{nullity}(T)$.
 42. For $T: M_{3,3} \rightarrow M_{3,3}$ and $\text{rank}(T) = 5$, find $\text{nullity}(T)$.

Finding a Power of a Standard Matrix In Exercises 43–46, find the specified power of A , the standard matrix for T .

43. $T: R^3 \rightarrow R^3$, reflection in the xy -plane. Find A^2 .
 44. $T: R^3 \rightarrow R^3$, projection onto the xy -plane. Find A^2 .
 45. $T: R^2 \rightarrow R^2$, counterclockwise rotation through the angle θ . Find A^3 .
 46. **Calculus** $T: P_3 \rightarrow P_3$, differential operator D_x . Find A^2 .

Finding Standard Matrices for Compositions In Exercises 47 and 48, find the standard matrices for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

47. $T_1: R^2 \rightarrow R^3$, $T_1(x, y) = (x, x + y, y)$
 $T_2: R^3 \rightarrow R^2$, $T_2(x, y, z) = (0, y)$
 48. $T_1: R \rightarrow R^2$, $T_1(x) = (x, 4x)$
 $T_2: R^2 \rightarrow R$, $T_2(x, y) = (y + 3x)$

Finding the Inverse of a Linear Transformation In Exercises 49–52, determine whether the linear transformation is invertible. If it is, find its inverse.

49. $T: R^2 \rightarrow R^2$, $T(x, y) = (0, y)$
 50. $T: R^2 \rightarrow R^2$,
 $T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$
 51. $T: R^2 \rightarrow R^2$, $T(x, y) = (x, -y)$
 52. $T: R^3 \rightarrow R^2$, $T(x, y, z) = (x + y, y - z)$

One-to-One, Onto, and Invertible Transformations In Exercises 53–56, determine whether the linear transformation represented by the matrix A is (a) one-to-one, (b) onto, and (c) invertible.

53. $A = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$ 54. $A = \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$
 55. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ 56. $A = \begin{bmatrix} 4 & 0 & 7 \\ 5 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Finding the Image Two Ways In Exercises 57 and 58, find $T(\mathbf{v})$ by using (a) the standard matrix and (b) the matrix relative to B and B' .

57. $T: R^2 \rightarrow R^3$,
 $T(x, y) = (-x, y, x + y)$, $\mathbf{v} = (0, 1)$,
 $B = \{(1, 1), (1, -1)\}$, $B' = \{(0, 1, 0), (0, 0, 1), (1, 0, 0)\}$
 58. $T: R^2 \rightarrow R^2$,
 $T(x, y) = (2y, 0)$, $\mathbf{v} = (-1, 3)$,
 $B = \{(2, 1), (-1, 0)\}$, $B' = \{(-1, 0), (2, 2)\}$

Finding a Matrix for a Linear Transformation In Exercises 59 and 60, find the matrix A' for T relative to the basis B' .

59. $T: R^2 \rightarrow R^2$, $T(x, y) = (x - 3y, y - x)$,
 $B' = \{(1, -1), (1, 1)\}$
 60. $T: R^3 \rightarrow R^3$, $T(x, y, z) = (x + 3y, 3x + y, -2z)$,
 $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

Similar Matrices In Exercises 61 and 62, use the matrix P to show that the matrices A and A' are similar.

61. $P = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $A = \begin{bmatrix} 6 & -3 \\ 2 & -2 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & -9 \\ -1 & 3 \end{bmatrix}$
 62. $P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

63. Define $T: R^3 \rightarrow R^3$ by $T(\mathbf{v}) = \text{proj}_{\mathbf{u}} \mathbf{v}$, where $\mathbf{u} = (0, 1, 2)$.
 (a) Find A , the standard matrix for T .
 (b) Let S be the linear transformation represented by $I - A$. Show that S is of the form
 $S(\mathbf{v}) = \text{proj}_{\mathbf{w}_1} \mathbf{v} + \text{proj}_{\mathbf{w}_2} \mathbf{v}$
 where $\mathbf{w}_1$ and $\mathbf{w}_2$ are fixed vectors in R^3 .
 (c) Show that the kernel of T is equal to the range of S .
 64. Define $T: R^2 \rightarrow R^2$ by $T(\mathbf{v}) = \text{proj}_{\mathbf{u}} \mathbf{v}$, where $\mathbf{u} = (4, 3)$.
 (a) Find A , the standard matrix for T , and show that $A^2 = A$.
 (b) Show that $(I - A)^2 = I - A$.
 (c) Find $A\mathbf{v}$ and $(I - A)\mathbf{v}$ for $\mathbf{v} = (5, 0)$.
 (d) Sketch the graphs of $\mathbf{u}$, $\mathbf{v}$, $A\mathbf{v}$, and $(I - A)\mathbf{v}$.
 65. Let S and T be linear transformations from V into W . Show that $S + T$ and kT are both linear transformations, where $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$ and $(kT)(\mathbf{v}) = kT(\mathbf{v})$.
 66. **Proof** Let $T: R^2 \rightarrow R^2$ such that $T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$, where A is a 2×2 matrix. (Such a transformation is called an **affine transformation**.) Prove that T is a linear transformation if and only if $\mathbf{b} = \mathbf{0}$.

Sum of Two Linear Transformations In Exercises 67 and 68, consider the sum $S + T$ of two linear transformations $S: V \rightarrow W$ and $T: V \rightarrow W$, defined as $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$.

67. **Proof** Prove that $\text{rank}(S + T) \leq \text{rank}(S) + \text{rank}(T)$.
 68. Give an example for each.
 (a) $\text{Rank}(S + T) = \text{rank}(S) + \text{rank}(T)$
 (b) $\text{Rank}(S + T) < \text{rank}(S) + \text{rank}(T)$
 69. **Proof** Let $T: P_3 \rightarrow R$ such that
 $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1 + a_2 + a_3$.
 (a) Prove that T is a linear transformation.
 (b) Find the rank and nullity of T .
 (c) Find a basis for the kernel of T .

70. Proof Let

$$T: V \rightarrow U \quad \text{and} \quad S: U \rightarrow W$$

be linear transformations.

(a) Prove that $\ker(T)$ is contained in $\ker(S \circ T)$.

(b) Prove that if $S \circ T$ is onto, then so is S .

71. Let V be an inner product space. For a fixed nonzero vector $\mathbf{v}_0$ in V , let $T: V \rightarrow R$ be the linear transformation $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle$. Find the kernel, range, rank, and nullity of T .

72. Calculus Let $B = \{1, x, \sin x, \cos x\}$ be a basis for a subspace W of the space of continuous functions, and let D_x be the differential operator on W . Find the matrix for D_x relative to the basis B . Find the range and kernel of D_x .

73. Writing Are the vector spaces R^4 , $M_{2,2}$, and $M_{1,4}$ exactly the same? Describe their similarities and differences.

74. Calculus Define $T: P_3 \rightarrow P_3$ by

$$T(p) = p(x) + p'(x).$$

Find the rank and nullity of T .

Identifying and Representing a Transformation In Exercises 75–80, (a) identify the transformation, and (b) graphically represent the transformation for an arbitrary vector in R^2 .

$$75. T(x, y) = (x, 2y) \qquad 76. T(x, y) = (x + y, y)$$

$$77. T(x, y) = (x, y + 3x) \qquad 78. T(x, y) = (5x, y)$$

$$79. T(x, y) = (x + 5y, y) \qquad 80. T(x, y) = \left(x, y + \frac{3}{2}x\right)$$

Sketching an Image of a Triangle In Exercises 81–84, sketch the image of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ under the specified transformation.

81. T is a reflection in the x -axis.

82. T is the expansion represented by $T(x, y) = (2x, y)$.

83. T is the shear represented by $T(x, y) = (x + 3y, y)$.

84. T is the shear represented by $T(x, y) = (x, y + 2x)$.

Giving a Geometric Description In Exercises 85 and 86, give a geometric description of the linear transformation defined by the matrix product.

$$85. \begin{bmatrix} 0 & 12 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$86. \begin{bmatrix} 1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Finding a Matrix to Produce a Rotation In Exercises 87–90, find the matrix that produces the rotation. Then find the image of the vector $(1, -1, 1)$.

87. 45° about the z -axis **88.** 90° about the x -axis

89. 60° about the x -axis **90.** 30° about the y -axis

Determining a Matrix to Produce a Pair of Rotations In Exercises 91–94, determine the matrix that produces the pair of rotations.

91. 60° about the x -axis and then 30° about the z -axis

92. 120° about the y -axis and then 45° about the z -axis

93. 30° about the y -axis and then 45° about the z -axis

94. 60° about the x -axis and then 60° about the z -axis

Finding an Image of a Unit Cube In Exercises 95–98, find the image of the unit cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$ when it rotates through the given angle.

95. 45° about the z -axis **96.** 90° about the x -axis

97. 30° about the x -axis **98.** 120° about the z -axis

True or False? In Exercises 99–102, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

99. (a) Reflections that map a point in the xy -plane to its mirror image across the line $y = x$ are linear transformations that are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Horizontal expansions or contractions are linear transformations that are defined by the matrix $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

100. (a) Reflections that map a point in the xy -plane to its mirror image across the x -axis are linear transformations that are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) Vertical expansions or contractions are linear transformations that are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

101. (a) In calculus, any linear function is also a linear transformation from R^2 to R^2 .

(b) A linear transformation is onto if and only if, for all $\mathbf{u}$ and $\mathbf{v}$ in V , $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

(c) For ease of computation, it is best to choose a basis for V such that the transformation matrix is diagonal.

102. (a) For polynomials, the differential operator D_x is a linear transformation from P_n into P_{n-1} .

(b) The set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{v}$ is the kernel of T .

(c) The standard matrix A of the composition of two linear transformations $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$ is the product of the standard matrix for T_2 and the standard matrix for T_1 .

6 Projects

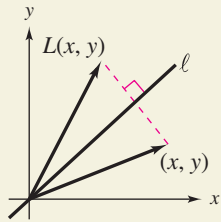


Figure 6.10

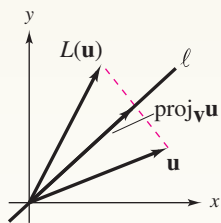


Figure 6.11

Let ℓ be the line $ax + by = 0$ in $\mathbb{R}^2$. The linear transformation $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps a point (x, y) to its mirror image with respect to ℓ is called the **reflection** in ℓ . (See Figure 6.10.) The goal of these two projects is to find the matrix for this reflection relative to the standard basis.

1 Reflections in $\mathbb{R}^2$ (I)

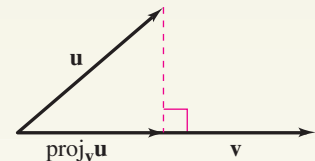
In this project, you will use transition matrices to determine the standard matrix for the reflection L in the line $ax + by = 0$.

1. Find the standard matrix for L for the line $x = 0$.
2. Find the standard matrix for L for the line $y = 0$.
3. Find the standard matrix for L for the line $x - y = 0$.
4. Consider the line ℓ represented by $x - 2y = 0$. Find a vector $\mathbf{v}$ parallel to ℓ and another vector $\mathbf{w}$ orthogonal to ℓ . Determine the matrix A for the reflection in ℓ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the reflection relative to the standard basis. Use this matrix to find the images of the points $(2, 1)$, $(-1, 2)$, and $(5, 0)$.
5. Consider the general line ℓ represented by $ax + by = 0$. Find a vector $\mathbf{v}$ parallel to ℓ and another vector $\mathbf{w}$ orthogonal to ℓ . Determine the matrix A for the reflection in ℓ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the reflection relative to the standard basis.
6. Find the standard matrix for the reflection in the line $3x + 4y = 0$. Use this matrix to find the images of the points $(3, 4)$, $(-4, 3)$, and $(0, 5)$.

2 Reflections in $\mathbb{R}^2$ (II)

In this project, you will use projections to determine the standard matrix for the reflection L in the line $ax + by = 0$. Recall that the projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$ (shown at the right) is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$



1. Find the standard matrix for the projection onto the y -axis. That is, find the standard matrix for $\text{proj}_{\mathbf{v}} \mathbf{u}$ when $\mathbf{v} = (0, 1)$.
2. Find the standard matrix for the projection onto the x -axis.
3. Consider the line ℓ represented by $x - 2y = 0$. Find a vector $\mathbf{v}$ parallel to ℓ and another vector $\mathbf{w}$ orthogonal to ℓ . Determine the matrix A for the projection onto ℓ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the projection relative to the standard basis. Use this matrix to find $\text{proj}_{\mathbf{v}} \mathbf{u}$ for $\mathbf{u} = (2, 1)$, $\mathbf{u} = (-1, 2)$, and $\mathbf{u} = (5, 0)$.
4. Consider the general line ℓ represented by $ax + by = 0$. Find a vector $\mathbf{v}$ parallel to ℓ and another vector $\mathbf{w}$ orthogonal to ℓ . Determine the matrix A for the projection onto ℓ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the projection relative to the standard basis.
5. Use Figure 6.11 to show that $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{1}{2}(\mathbf{u} + L(\mathbf{u}))$, where L is the reflection in the line ℓ . Solve this equation for L and compare your answer with the formula from the first project.

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